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Yang-Mills Solutions in Heterotic Flux Compactifications

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Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen benutzt habe.

Felix Lubbe

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Contents

Chapter 1

Introduction

One major motivation to search for "more fundamental" theories than the standard model is that it incorporates only three of the four known fundamental forces, leaving out gravity. At the level of small distances, the predictions of the standard model were precisely verified. On the other hand there is a theory which is equally well verified on large scales and which describes gravity: The *Theory of General Relativity*. However, every attempt to unify these two theories proved yet unsuccessful.

Currently there are a few approaches to unify these two, which are to be investigated. Among them are the so-called *String Theories*, and as a limiting case of these the *supergravity theories*.

String Theory. The basic idea in String Theory is to replace the point-like particles (as they are e.g. described by the standard model) by 1-dimensional oscillating strings. The different modes of a string then correspond to the different particles one observes. The theory itself fixes all but one parameter, which is a desirable feature. However, consistency requires a formulation in $d = 26$ dimensions [36, 37].

This 26-dimensional theory is not able to describe the observed particles correctly, as it only features one type of them, the bosons. Therefore it is also called the *Bosonic String Theory*. Apart from only having a single particle type, it also has the major flaw of postulating a particle with imaginary mass, the *tachyon*.

One gets rid of these drawbacks by incorporating supersymmetry. This also reduces the critical dimension from 26 to 10. The resulting theories now feature both, bosonic and fermionic, degrees of freedom, and at the same time they do not include tachyons. In the low-energy limit, two of them (heterotic and type I) include the supersymmetric Yang-Mills theory.

One problem with Superstring Theory is that it requires 10 dimensions, though space-time appears to be 4-dimensional. A way out of this is to compactify the additional 6 dimensions, effectively generating a space-time $\mathcal{M}^4 \times X^6$, where X^6 is a 6-dimensional compact space and $\mathcal{M}^4$ the observable space-time.

Demanding at least one space-time supersymmetry on $\mathcal{M}^4$ and the absence of higher-form fields led to the compact X^6 to be a Calabi-Yau manifold [24], which is a 6-dimensional manifold with holonomy group $SU(3)$. These demands were quite strong and produced theories with particle spectra not matching the standard model. The Calabi-Yau condition is generalized by so-called flux compactifications, by allowing fluxes from higher-form fields [25, 26]. A particular such geometry for X^6 is the nearly Kähler one. The only known examples of compact nearly Kähler 6-manifolds are three non-symmetric coset spaces and a product space,

$$\frac{G_2}{SU(3)}, \quad \frac{SU(3)}{U(1)\times U(1)}, \quad \frac{Sp(2)}{Sp(1)\times U(1)}, \quad S^3\times S^3,$$

which were classified in [23].

Supergravity. Instead of trying to quantize pure gravity, one may also try to incorporate supersymmetry into general relativity. The aim of such efforts is to improve the quantization behaviour for the resulting theory [29]. This so-called supergravity theory is a field theory, which possesses supersymmetry as a local symmetry [28]. Although initially developed in $d = 4$ dimensions, the theory also exists in other dimensions, for example in $d = 10$ and $d = 11$. The supergravity theory in $d = 11$ dimensions was the first candidate of a "theory of everything".

Considering supergravity theories with $d > 4$, one again arrives at studying dimensional reductions and compactifications. Furthermore, all 10-dimensional supergravity theories are limiting cases of 10-dimensional String Theories at low energies [31].

Outline. In this thesis we concentrate on heterotic supergravity, compactified on $\mathcal{M}^4 \times X^6$ with X^6 from the list given above [30, 32]. A part of heterotic supergravity is Yang-Mills theory. We investigate the Yang-Mills equations for the gauge group G on the coset spaces G/H, with G/H being $SU(3)/U(1)\times U(1)$ or $Sp(2)/Sp(1)\times U(1)$. In addition, we study their analytical and numerical solutions on the space $\mathbb{R}\times G/H$. By identification of the fields occuring in the Yang-Mills equations, we also get a reduction to the case $G_2/SU(3)$, which was discussed in [9].

The thesis is organized as follows. Since we are dealing with coset spaces G/H, they are introduced in **chapter 2**. These spaces are nearly Kähler manifolds, which are almost complex manifolds, as introduced in **chapter 3**.

As mentioned above, we want to get from the coset space G/H to the product space $\mathbb{R}\times G/H$. This space has a geometry induced from G/H, which will be described in **chapter 4**.

In **chapter 5**, Yang-Mills theories on principal bundles are introduced. Finally, in the **chapters 6 and 7**, this theory is considered on the spaces G/H and $\mathbb{R}\times G/H$. Some of the results presented here have appeared in [10].

Mathematical background material is collected in the **appendix**.

Chapter 2

Homogeneous Spaces

In this chapter we introduce the so-called *homogeneous spaces*. They turn out to be coset spaces which admit a set of invariant objects, making them "look the same" at every point. This chapter is based on [41, 48, 50, 51].

2.1 Coset Spaces

Definition 2.1.1. Let M be a manifold and G a Lie group. G acts on M from the left, if there is a smooth map

$$\phi : G \times M \to M, \qquad (g, x) \mapsto g \cdot x \tag{2.1}$$

which has the following properties:

(i) $l_g : x \in M \mapsto g \cdot x \in M$ is a diffeomorphism for every $g \in G$.

(ii) $x = e \cdot x$ for every $x \in M$, where e is the identity element in G.

(iii) $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ for all $x \in M$, $g, h \in G$.

Given such an action of G on M, the pair (M, G) is called a *Lie transformation group*.

Definition 2.1.2 (Homogeneous Space)**.** Let M be a manifold and G a Lie group. If G acts transitively on M, then M is called a *homogeneous space*.

Example 2.1.1. (i) We consider the Lie group $GL(n, \mathbb{R})$. $GL(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n \setminus \{0\}$ via $\phi(A, x) := A \cdot x$.

(ii) The group $O(n, \mathbb{R})$ acts transitively on the sphere $S^{n-1}(r) := \{x \in \mathbb{R}^n : \|x\| = r\}$, but not on $\mathbb{R}^n \setminus \{0\}$.

Let G be a Lie group and H be a closed subgroup. Then the quotient space G/H is a homogeneous space:

Theorem 2.1.1. *Let H be a closed subgroup of a Lie group G. Then there exists a manifold structure on the quotient space G/H, such that:*

(i) The projection $\pi : G \to G/H$ is smooth.

(ii) Given the left action $(g, a) \in G \times G/H \mapsto ga \in G/H$, G/H is a homogeneous space.

(iii) *For every equivalence class $a \in G/H$ there exists a neighborhood $W(a) \subset G/H$ and a smooth map $s_a : W(a) \to G$, such that $\pi \circ s_a = \mathrm{id}_{W(a)}$.*

In fact every homogeneous space (M, G) is a quotient space $M = G/H$ for a suitable closed subgroup H. This means, that theorem B.5.2 and remark B.5.1 also hold for homogeneous spaces.

Theorem 2.1.2. *Let G be a Lie group, which acts on M from the left and $x \in M$. Then*

$$\Psi : G/G_x \to M, \quad a \mapsto ax \tag{2.2}$$

is a G-equivariant diffeomorphism, i.e. $\Psi(ga) = g\Psi(a)$ for all $g \in G$, $a \in G/G_x$.

Example 2.1.2. We consider the sphere $S^n := S^n(1)$ from the previous example. The special orthogonal group $SO(n+1)$ acts transitively on S^n. The stabilizer group of $e_{n+1} := (0, \dots, 0, 1)^T$ is the subgroup $SO(n) \hookrightarrow SO(n+1)$, embedded via

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

So we can represent S^n as a homogeneous space, $S^n = SO(n+1)/SO(n)$.

Definition 2.1.3. Let $H \subset G$ be a closed subgroup of a Lie group, $\mathfrak{h}$ and $\mathfrak{g}$ the corresponding Lie algebras. Then the homogeneous space G/H is called *reductive*, if there exists a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, such that

$$\mathrm{Ad}\,(H)\mathfrak{m} \subset \mathfrak{m}. \tag{2.3}$$

Remark 2.1.1. Equation (2.3) implies

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}. \tag{2.4}$$

If H is connected, then also the converse is true.

Example 2.1.3. (i) $S^n = SO(n+1)/SO(n)$ is a reductive representation of the sphere as a homogeneous space.

(ii) $SU(3)/U(1) \times U(1)$: We choose generators $\{I_k\}_{k=1,\dots,8}$ of $\mathfrak{su}(3)$, which together with the structure constants are given in appendix D.2. Then the $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$-subalgebra is embedded via the generators I_7 and I_8. The corresponding decomposition

$$\mathfrak{su}(3) = (\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus \mathfrak{m}, \tag{2.5}$$

where $\mathfrak{m}$ is generated by $(I_k)_{k=1,\dots,6}$, fulfills $[\mathfrak{u}(1) \times \mathfrak{u}(1), \mathfrak{m}] \subset \mathfrak{m}$.

(iii) $Sp(2)/Sp(1) \times U(1)$: We choose generators $\{I_k\}_{k=1,\dots,10}$ of $\mathfrak{sp}(2)$, which together with the structure constants are given in appendix D.3. Then we embed $\mathfrak{sp}(1)$ via I_7, I_8, I_9 and $\mathfrak{u}(1)$ via I_{10}. This choice corresponds to a non-maximal embedding of the subgroup [16]. We arrive at the decomposition

$$\mathfrak{sp}(2) = (\mathfrak{sp}(1) \oplus \mathfrak{u}(1)) \oplus \mathfrak{m}, \tag{2.6}$$

where $\mathfrak{m}$ is generated by $(I_k)_{k=1,\dots,6}$. Then with $\mathfrak{h} := \mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ the condition $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ holds, meaning $Sp(2)/Sp(1) \times U(1)$ is reductive.

Remark 2.1.2 (Coordinate representation)**.** If we introduce coordinates $\{I_A\}_{A=1,\dots,n}$ on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $n := \dim \mathfrak{g}$, $m := \dim \mathfrak{m}$, we can write for a general Lie algebra decomposition, where $\mathfrak{h}$ is a subalgebra,

$$\begin{aligned}
[I_i, I_j] &= \sum_{k=m+1}^{n} f_{ij}{}^{k} I_k, \\
[I_i, I_a] &= \sum_{b=1}^{m} f_{ia}{}^{b} I_b + \sum_{k=m+1}^{n} f_{ia}{}^{k} I_k, \\
[I_a, I_b] &= \sum_{c=1}^{m} f_{ab}{}^{c} I_c + \sum_{k=m+1}^{n} f_{ab}{}^{k} I_k.
\end{aligned} \tag{2.7}$$

If $\mathfrak{g}$ is reductive, then we have $f_{ia}{}^{b} = 0$ for all $i = m+1, \dots, n$ and $a, b = 1, \dots, m$, so that we can write (2.4) as

$$\begin{aligned}
&[I_i, I_j] = \sum_{k=m+1}^{n} f_{ij}{}^{k} I_k, \qquad [I_i, I_a] = \sum_{b=1}^{m} f_{ia}{}^{b} I_b, \\
&[I_a, I_b] = \sum_{c=1}^{m} f_{ab}{}^{c} I_c + \sum_{k=m+1}^{n} f_{ab}{}^{k} I_k.
\end{aligned} \tag{2.8}$$

The index ranges for the above equations are given by

$$\begin{aligned}
A, B, C, \dots &\in \{1, \dots, \dim(G)\}, \\
a, b, c, \dots &\in \{1, \dots, \dim(G/H)\}, \\
i, j, k, \dots &\in \{\dim(G/H) + 1, \dots, \dim(G)\}.
\end{aligned} \tag{2.9}$$

2.2 Invariant Forms on Homogeneous Spaces

Remark 2.2.1. Let G be a Lie group, H be a closed subgroup and G/H be the corresponding homogeneous space. Let further $\widetilde{H}$ be the linear isotropy group at the origin $e \in G/H$, which is the point represented by the coset $H = e$. Then $\widetilde{H}$ is a group of linear transformations of the tangent space $T_e(G/H)$, each induced induced by an element of H which leaves the point e fixed.

Since H is compact, so is $\widetilde{H}$. Further there is a positive definite inner product g_e on $T_e(G/H)$, which is invariant by $\widetilde{H}$. Now for each $x \in G/H$, we take an element $a \in G$ such that $a(e) = x$ and define an inner product g_x on $T_x(G/H)$ by

$$g_x(X, Y) = g_e(a^{-1}X, a^{-1}Y), \qquad \forall X, Y \in T_x(G/H). \tag{2.10}$$

Then g_x is independent of the choice of $a \in G$ and the Riemannian metric thus obtained is invariant by G.

Definition 2.2.1. A homogeneous space G/H provided with an invariant Riemannian metric is called a *Riemannian homogeneous space.*

Proposition 2.2.1. *Let $M = G/H$ be a homogeneous space and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ its Lie algebra decomposition, and $e \in M$ the identity. Then there is a natural one-to-one correspondence between G-invariant indefinite Riemannian metrics g on M and the* $\mathrm{Ad}\,(H)$*-invariant non-degenerate symmetric bilinear forms B on $\mathfrak{m}$. The correspondence is given by*

$$B(\widetilde{X}, \widetilde{Y}) = g(X, Y)_e, \qquad \forall X, Y \in \mathfrak{g}, \tag{2.11}$$

where $\widetilde{X}$, $\widetilde{Y}$ are the elements of $\mathfrak{m}$ represented by X and Y respectively. A form B is positive definite if and only if the corresponding metric g is positive definite.

Corollary 2.2.1. *If $M = G/H$ is reductive with an* $\mathrm{Ad}\,(H)$*-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, then there is a natural one-to-one correspondence between the G-invariant indefinite Riemannian metrics g on M and the* $\mathrm{Ad}\,(H)$*-invariant non-degenerate symmetric bilinear forms B on $\mathfrak{m}$. The correspondence is given by*

$$B(X, Y) = g(X, Y)_e, \qquad \forall X, Y \in \mathfrak{m}. \tag{2.12}$$

Now for a manifold M of dimension n, let G be a Lie subgroup of $GL(n, \mathbb{R})$. A G-structure on M is a principal subbundle P of the bundle F of linear frames (cf. C.1.3) with structure group G.

Theorem 2.2.1. *Let P be an K-invariant G-structure on a reductive homogeneous space $M = K/H$ with decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$. Then there is a one-to-one correspondence between the set of K-invariant connections on P and the set of linear mappings $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{g}$ such that*

$$\Lambda_{\mathfrak{m}}(\mathrm{Ad}\,(h)(X)) = \mathrm{Ad}\,(\lambda(h))(\Lambda_{\mathfrak{m}}(X)), \qquad X \in \mathfrak{m},\ h \in H, \tag{2.13}$$

where λ denotes the linear isotropy representation[1] *$H \to G$. The correspondence is given by*

$$\Lambda(X) = \begin{cases} \lambda(X), & \text{if } X \in \mathfrak{h}, \\ \Lambda_{\mathfrak{m}}(X), & \text{if } X \in \mathfrak{m}. \end{cases} \tag{2.14}$$

[1] The *linear isotropy representation* is the homomorphism of H into the group of linear transformations of $T_e(G/H)$ which assigns to each $h \in H$ the differential of h at e.

Chapter 3

Complex Geometry

A natural extension to the concept of real manifolds are *(almost) complex manifolds*, which locally look like the space $\mathbb{C}^n$. These manifolds carry an additional so-called *almost complex structure*, which acts on the tangent spaces. In a chart it roughly corresponds to a multiplication with the imaginary unit. The literature for this chapter is mainly [41], but also [48–50].

3.1 Almost Complex Structures, Tensor Decomposition and Exterior Forms

Definition 3.1.1 (Almost Complex Structure)**.** Let M be a real manifold of dimension $2m$. We define an *almost complex structure* J on M to be a smooth tensor field J on M, which is an endomorphism of T_pM and satisfies $J_p^2 = -\mathrm{id}_p$ at every point $p \in M$. A manifold with almost complex structure is called *almost complex manifold*.

Remark 3.1.1. Choosing coordinates $\{\partial_{x^1}, \ldots, \partial_{x^{2m}}\}$ on T_pM, we may write the above condition as

$$\sum_{b=1}^{2m} J_a{}^b J_b{}^c = -\delta_a{}^c. \tag{3.1}$$

Let $v = \sum_{a=1}^{2m} v^a \frac{\partial}{\partial x^a}$ be a smooth vector field on M. Then we define a new vector field Jv by $(Jv)^b := \sum_{a=1}^{2m} J_a{}^b v^a$. Thus J acts linearly on vector fields. From the definition we get $J(Jv) = -v$. J gives each tangent space T_pM the structure of a complex vector space.

Definition 3.1.2. For all smooth vector fields v, w on M, we define a vector field $N_J(v, w)$ by

$$N_J(v, w) := [v, w] + J\left([Jv, w] + [v, Jw]\right) - [Jv, Jw], \tag{3.2}$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. One can show that N_J is a tensor, meaning that $N_J(v, w)$ is pointwise bilinear in v and w. N_J is called the *Nijenhuis tensor of J*.

Definition 3.1.3 (Complex Manifold)**.** Let M be a real manifold of dimension $2m$, and J an almost complex structure on M. We call J a *complex structure*, if $N_J \equiv 0$ on M. A *complex manifold* is a manifold M equipped with a complex structure J. We use the notation (M, J) to refer to a manifold and its complex structure.

In order to construct an example, we give another (equivalent) definition.

Definition 3.1.4 (Complex Manifold)**.** Let M be a real manifold of dimension $2m$. A *complex chart* on M is a pair (U, ψ), where U is open in M and $\psi : U \to \mathbb{C}^m$ is a diffeomorphism between U and some open subset of $\mathbb{C}^m$. If (U_1, ψ_1) and (U_2, ψ_2) are two complex charts, then the *transition function* is $\psi_{12} : \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$, given by $\psi_{12} := \psi_2 \circ \psi_1^{-1}$. We say M is a *complex manifold*, if it has an atlas of complex charts (U, ψ), such that all transition functions are holomorphic, as maps from $\mathbb{C}^m$ to itself.

Example 3.1.1 (Complex Projective Space)**.** The *complex projective space* can be defined as

$$\mathbb{CP}^m := \frac{U(m+1)}{U(m) \times U(1)}. \tag{3.3}$$

Since $S^{2m+1} = U(m+1)/U(m)$, $\mathbb{CP}^m$ may be thought of as the set of one-dimensional subspaces of $\mathbb{C}^{m+1}$. We further consider charts $U_a \subset \mathbb{C}^{m+1} \setminus \{0\}$ where $z^a \neq 0$. In U_a we define the *inhomogeneous coordinates*

$$\xi^{\mu}_{(a)} := \frac{z^a}{z^{\mu}}, \qquad k \neq a. \tag{3.4}$$

Then in $U_a \cap U_b$, a coordinate transformation is given by

$$\psi_{ab} : \mathbb{C}^m \to \mathbb{C}^m, \qquad \xi^{\mu}_{(b)} \mapsto \xi^{l}_{(a)} := \frac{z^{\mu}}{z^{l}} \xi^{\mu}_{(b)}, \tag{3.5}$$

which is holomorphic.

The analogue of a smooth vector bundle over real manifold is a holomorphic vector bundle, which we define in the following

Definition 3.1.5 (Holomorphic Vector Bundle)**.** Let M be a complex manifold. Let $\{E_p : p \in M\}$ be a family of complex vector spaces of dimension k, parameterized by M. Let E be the total space of this family, and $\pi : E \to M$ be the natural projection. Suppose also that E has the structure of a complex manifold. Then E is called a *holomorphic vector bundle* with fibre $\mathbb{C}^k$, if the following conditions hold.

(i) The map $\pi : E \to M$ is a holomorphic map of complex manifolds.

(ii) For each $p \in M$ there exists an open neighborhood $U \subset M$, and a biholomorphic map $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$.

(iii) In part (ii), for each $u \in U$ the map ϕ_U takes E_u to $\{u\} \times \mathbb{C}^k$, and this is an isomorphism between E_u and $\mathbb{C}^k$ as complex vector spaces.

The vector space E_p is called the *fibre* of E over p.[1]

[1] One may also refer to E as the holomorphic vector bundle, implicitly assuming the rest of the structure is given.

Definition 3.1.6 (Holomorphic Section)**.** Let E be a holomorphic vector bundle over M, with projection $\pi : E \to M$. A *holomorphic section* s of E is a holomorphic map $s : M \to E$, such that $\pi \circ s$ is the identity map on M.

Remark 3.1.2 (Vector Field Decomposition)**.** Let (M, J) be a complex manifold.

(i) A vector field on M is a section in the (complex) tangent bundle TM.

(ii) Let $p \in M$ and

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^m} \right\} \tag{3.6}$$

be a basis for T_pM. Then we define a basis with $2m$ vectors,

$$\frac{\partial}{\partial z^\mu} := \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - \mathrm{i}\frac{\partial}{\partial y^\mu} \right), \qquad \frac{\partial}{\partial \bar{z}^\mu} := \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} + \mathrm{i}\frac{\partial}{\partial y^\mu} \right) \tag{3.7}$$

with $\mu = 1, \dots, m$. We now choose a basis in such a way that the complex structure J acts on it as follows:

$$J_p\left(\frac{\partial}{\partial x^\mu} \right) = \frac{\partial}{\partial y^\mu}, \qquad J_p\left(\frac{\partial}{\partial y^\mu} \right) = -\frac{\partial}{\partial x^\mu}, \qquad \forall \mu = 1, \dots, m. \tag{3.8}$$

J_p may be extended on the complexified tangent space $T_pM \otimes \mathbb{C}$,

$$J_p(X + \mathrm{i}Y) := J_pX + \mathrm{i}J_pY, \tag{3.9}$$

i.e. in the basis (3.7)

$$J_p\left(\frac{\partial}{\partial z^\mu} \right) = \mathrm{i}\frac{\partial}{\partial z^\mu} \qquad \text{and} \qquad J_p\left(\frac{\partial}{\partial \bar{z}^\mu} \right) = -\mathrm{i}\frac{\partial}{\partial \bar{z}^\mu}, \tag{3.10}$$

thus the action of J_p roughly corresponds to the multiplication with $\pm\mathrm{i}$ in a chart. This means we may write J_p as

$$J_p = \sum_{\mu=1}^{m} \left(\mathrm{i}\,\mathrm{d}z^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\,\mathrm{d}\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu} \right), \tag{3.11}$$

where $\mathrm{d}z^\mu$, $\mathrm{d}\bar{z}^\mu$ are dual to $\partial/\partial z^\mu$ respectively $\partial/\partial \bar{z}^\mu$. Now with respect to (3.6) and (3.8) one may regard J as a real-valued matrix with dimension $2m$,

$$J_p = \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix}, \tag{3.12}$$

or in the basis (3.7) as a complex-valued matrix with dimension $2m$, given by

$$J_p = \begin{pmatrix} \mathrm{i}\mathbf{1}_m & \mathbf{0}_m \\ \mathbf{0}_m & -\mathrm{i}\mathbf{1}_m \end{pmatrix}. \tag{3.13}$$

We define the subspaces

$$T_pM^{\pm} := \{Z \in T_pM \otimes \mathbb{C} : J_p(Z) = \pm \mathrm{i} Z\}. \tag{3.14}$$

Then we have $T_pM \otimes \mathbb{C} = T_pM^+ \oplus T_pM^-$, and call $Z^{\pm} \in T_p^{\pm}M$ a *holomorphic* resp. *antiholomorphic vector*, and the corresponding bases (3.7) are called *holomorphic* resp. *antiholomorphic basis*. This definition is extended pointwise to vector fields and we have $T_pM^- = \overline{T_pM^+}$ as well as the decomposition $\mathfrak{X}(M) \otimes \mathbb{C} = \mathfrak{X}(M)^+ \oplus \mathfrak{X}(M)^-$ with the same notion as above.

(iii) Let (M, J) be an almost complex manifold. Then $J_p^2 = -\mathrm{id}_p$ on T_pM, and we may extend J_p to $T_pM \otimes \mathbb{C}$. Now we can adapt the above classification scheme for vector fields. Note that there does not necessarily exist a basis in the form $\{\partial/\partial z^{\mu}\}_{\mu=1,\dots,m}$.

Definition 3.1.7. Let M be a differentiable manifold with $\dim_{\mathbb{R}} M = m$. Let $\omega, \eta \in \Omega_p^q(M)$ be two q-forms at $p \in M$. Then we define a *complex q-form* $\xi := \omega + \mathrm{i}\eta$. We denote the vector space of complex q-forms at p with $\Omega_p^q(M) \otimes \mathbb{C}$. Then $\Omega_p^q(M) \subset \Omega_p^q(M) \otimes \mathbb{C}$ and we define the *conjugate* of ξ as $\bar{\xi} := \omega - \mathrm{i}\eta$. Further we call ξ *real*, if $\bar{\xi} = \xi$.

Definition 3.1.8. Let M be an (almost) complex manifold with complex dimension m. Let $\omega \in \Omega_p^q(M) \otimes \mathbb{C}$, $q \leq 2m$, and r, s be positive integers such that $r + s = q$. Let $V_i \in T_pM \otimes \mathbb{C}$ $(1 \leq i \leq q)$ be vectors in either T_pM^+ or T_pM^-. If $\omega(V_1, \dots, V_q) = 0$ unless r of the V_i are in T_pM^+ and s of the V_i are in T_pM^-, ω is said to be of *bidegree* (r, s) or simply an (r, s)-form. The set of (r, s)-forms at p is denoted by $\Omega_p^{r,s}(M)$. If an (r, s)-form is assigned smoothly at each point of M, we have an (r, s)-form defined over M. The set of (r, s)-forms over M is denoted by $\Omega^{r,s}(M)$.

Remark 3.1.3 (Interior Product)**.** We consider $\omega \in \Omega^{r,s}(M)$ on an (almost) complex manifold M with complex structure J, and $v, v_1, \dots, v_{r+s-1} \in T_pM \otimes \mathbb{C}$ at $p \in M$. Then v is decomposed into $v = v^+ + v^-$, with $v^{\pm} \in T_pM^{\pm}$. We define the *interior product* as

$$(v \lrcorner \omega)(v_1, \dots, v_{r+s-1}) := \omega(v, v_1, \dots, v_{r+s-1}). \tag{3.15}$$

Choosing a basis $\{\partial_{z^{\mu}}, \partial_{\bar{z}^{\mu}}\}$ on $T_pM^+ \oplus T_pM^-$, we write

$$\omega^a(v_1, \dots, v_{r+s-1}) := \left(\frac{\partial}{\partial z^a} \lrcorner \omega\right)(v_1, \dots, v_{r+s-1}), \tag{3.16}$$

$$\omega^{\bar{a}}(v_1, \dots, v_{r+s-1}) := \left(\frac{\partial}{\partial \bar{z}^a} \lrcorner \omega\right)(v_1, \dots, v_{r+s-1}), \tag{3.17}$$

and analogously for interior products of higher order, tensors $S \in T^{p,0}M$ and $T \in T^{p,q}M$.

Remark 3.1.4 (Complex Tensor Decomposition)**.** With the same setup as in the above remark, consider a tensor $T \in \Omega^{r,s}(M)$, with components $T^{a_1,\dots,a_r}_{b_1,\dots,b_s}$. The indices range over barred and non-barred components. Then we define tensors, labelled by greek indices $\alpha, \beta, \ldots$ and their conjugates $\bar{\alpha}, \bar{\beta}, \ldots$, by

$$(T^{\alpha})^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} := \frac{1}{2}\left(T^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} - \mathrm{i}\sum_{k=1}^{m} J^{a_1}_{k} T^{k,a_2,\ldots,a_r}_{b_1,\ldots,b_s}\right), \tag{3.18}$$

$$(T^{\bar{\alpha}})^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} := \frac{1}{2}\left(T^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} + \mathrm{i}\sum_{k=1}^{m} J^{a_1}_{k} T^{k,a_2,\ldots,a_r}_{b_1,\ldots,b_s}\right), \tag{3.19}$$

$$(T_{\beta})^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} := \frac{1}{2}\left(T^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} - \mathrm{i}\sum_{k=1}^{m} J^{k}_{b_1} T^{a_1,\ldots,a_r}_{k,b_2,\ldots,b_s}\right), \tag{3.20}$$

$$(T_{\bar{\beta}})^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} := \frac{1}{2}\left(T^{a_1,\ldots,a_r}_{b_1,\ldots,b_s} + \mathrm{i}\sum_{k=1}^{m} J^{k}_{a_1} T^{a_1,\ldots,a_r}_{\beta,b_2,\ldots,b_s}\right). \tag{3.21}$$

We refer to the components of these tensors by using greek letters which correspond to the latin indices, e.g. we will write T^{α} to refer to the component $(T^{\alpha})^{a}$ of a tensor $T \in \Omega^{1,0}(M)$. Further these operations are projections and satisfy

$$T^{a} = T^{\alpha} + T^{\bar{\alpha}}, \qquad T_{a} = T_{\alpha} + T_{\bar{\alpha}}. \tag{3.22}$$

We compute the exterior derivative of an (r,s)-form ω on a complex manifold. Choosing complex coordinates $\{z^{\mu}\}$ on $U \subset M$, ω can be written as

$$\omega = \frac{1}{r!s!}\sum_{\substack{\alpha_1,\ldots,\alpha_r=1\\ \beta_1,\ldots,\beta_s=1}}^{m} \omega_{\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s}\mathrm{d}z^{\alpha_1}\wedge\cdots\wedge\mathrm{d}z^{\alpha_r}\wedge\mathrm{d}\bar{z}^{\beta_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{\beta_s}. \tag{3.23}$$

Then we have

$$\begin{aligned}\mathrm{d}\omega = \frac{1}{r!s!}\sum_{\substack{\alpha_1,\ldots,\alpha_r=1\\ \beta_1,\ldots,\beta_s=1\\ \gamma=1}}^{m} &\left(\frac{\partial}{\partial z^{\gamma}}\omega_{\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s}\mathrm{d}z^{\gamma} + \frac{\partial}{\partial \bar{z}^{\gamma}}\omega_{\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s}\mathrm{d}\bar{z}^{\gamma}\right)\times\\ &\times \mathrm{d}z^{\alpha_1}\wedge\cdots\wedge\mathrm{d}z^{\alpha_r}\wedge\mathrm{d}\bar{z}^{\beta_1}\wedge\cdots\wedge\mathrm{d}\bar{z}^{\beta_s},\end{aligned} \tag{3.24}$$

i.e. $\mathrm{d}\omega$ is a mixture of an $(r+1,s)$-form and an $(r,s+1)$-form.

To generalize this calculation, let M now be an almost complex manifold. We define the projection

$$\pi^{r,s} : E^{k}(M) \to \Omega^{r,s}(M), \tag{3.25}$$

where $E^{k}(M)$ denotes the direct sum decomposition

$$E^{k}(M) := \bigoplus_{r+s=k}\Omega^{r,s}(M). \tag{3.26}$$

Then we can decompose the exterior derivative as

$$\mathrm{d}_{|\Omega^{r,s}(M)} = \partial + \bar{\partial}, \tag{3.27}$$

with

$$\begin{aligned} \partial &:= \pi^{r+1,s} \circ \mathrm{d}_{|\Omega^{r,s}(M)} : \Omega^{r,s} \to \Omega^{r+1,s}, \\ \bar{\partial} &:= \pi^{r,s+1} \circ \mathrm{d}_{|\Omega^{r,s}(M)} : \Omega^{r,s} \to \Omega^{r,s+1}. \end{aligned} \tag{3.28}$$

Definition 3.1.9. The operators ∂ and $\bar{\partial}$ in (3.28) are called the *Dolbeault operators.* We define an operator $\mathrm{d}^c : \Omega^k(M) \to \Omega^{k+1}(M)$ by $\mathrm{d}^c := \mathrm{i}(\bar{\partial} - \partial)$.

Remark 3.1.5. The identity $\mathrm{d}^2 = 0$ implies that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. Further the following identities hold:

$$\mathrm{d}\mathrm{d}^c + \mathrm{d}^c\mathrm{d} = 0, \quad (\mathrm{d}^c)^2 = 0, \quad \mathrm{d}\mathrm{d}^c = 2\mathrm{i}\partial\bar{\partial}. \tag{3.29}$$

Remark 3.1.6 (Holomorphic Forms)**.** Let $s \in \mathscr{C}^\infty(\Lambda^{p,0}M)$, so that s is a smooth section of $\Lambda^{p,0}M$. It can be shown that s is a holomorphic section of $\Lambda^{p,0}M$ if and only if $\bar{\partial}s = 0$ in $\mathscr{C}^\infty(\Lambda^{p,1}M)$. A holomorphic section of $\Lambda^{p,0}M$ is called a *holomorphic p-form.*

Definition 3.1.10 (Dolbeault Cohomology Group)**.** We define the *Dolbeault cohomology groups* $H^{p,q}_{\bar{\partial}}(M)$ of a complex manifold M by

$$H^{p,q}_{\bar{\partial}}(M) := \frac{\operatorname{Ker}\left(\bar{\partial} : \mathscr{C}^\infty(\Lambda^{p,q}M) \to \mathscr{C}^\infty(\Lambda^{p,q+1}M)\right)}{\operatorname{Im}\left(\bar{\partial} : \mathscr{C}^\infty(\Lambda^{p-1,q}M) \to \mathscr{C}^\infty(\Lambda^{p,q}M)\right)}. \tag{3.30}$$

Then the Dolbeault cohomology group $H^{p,0}_{\bar{\partial}}(M)$ is the vector space of holomorphic p-forms on M.

3.2 Hermitian Metrics, Kähler Metrics and Kähler Potentials

We begin by defining a metric on an almost complex manifold, which is compatible with the complex structure.

Definition 3.2.1 (Hermitian Metric)**.** Let M be a manifold with almost complex structure J, and let g be a Riemannian metric on M. We call g a *Hermitian metric* if

$$g(v, w) = g(Jv, Jw) \quad \forall v, w \in \mathfrak{X}(M). \tag{3.31}$$

An almost complex manifold (resp. a complex manifold) with a Hermitian metric is called an *almost Hermitian manifold* (resp. a *Hermitian manifold*).

Definition 3.2.2. Let (M, J) be an almost complex manifold and g a Hermitian metric. Let $\omega \in \Omega^2(M)$ be a 2-form that satisfies

$$\omega(v, w) = g(Jv, w), \qquad \forall v, w \in \mathfrak{X}(M). \tag{3.32}$$

Then ω is called the *Hermitian form* or *fundamental 2-form* on M.

Remark 3.2.1. The first definition is equivalent to demanding

$$g_{ab} = \sum_{c,d=1}^{m} J_a{}^c J_b{}^d g_{cd} \tag{3.33}$$

for the components of the metric g. The compatibility condition from the second definition is equivalent to

$$\omega_{ac} = \sum_{b=1}^{m} J_a{}^b g_{bc}. \tag{3.34}$$

Further ω is a $(1,1)$-form, and one may reconstruct g from ω using $g(v,w) = \omega(v, Jw)$. Therefore we call a $(1,1)$-form ω on a complex manifold *positive*, if $\omega(v, Jv) > 0$ for $v \neq 0$.

Definition 3.2.3 (Kähler Metric)**.** Let M be a almost complex manifold, and g a Hermitian metric on M, with Hermitian form ω. We say g is a *Kähler metric*, if $\mathrm{d}\omega = 0$. An almost complex manifold (resp. a complex manifold) with a Kähler metric is called an *almost Kähler manifold* (resp. a *Kähler manifold*). In this case we call ω the *almost Kähler form* (resp. *Kähler form*).

We give a few important facts about Kähler metrics, which especially contain a connection to G-structures on the corresponding manifolds.

Proposition 3.2.1. *Let M be a manifold of dimension $2m$, J an almost complex structure on M, and g a Hermitian metric, with Hermitian form ω. Let ∇ be the Levi-Civita connection of g. Then the following conditions are equivalent:*

(i) *J is a complex structure and g is Kähler,*

(ii) $\nabla J = 0$,

(iii) $\nabla \omega = 0$,

(iv) *The holonomy group of g is contained in $U(m)$, and J is associated to the corresponding $U(m)$-structure.*

On a compact Kähler manifold M we may find a real function ϕ, such that $\omega = \mathrm{d}\mathrm{d}^c\phi$. If M is a complex manifold, but not necessarily Kähler, the following result holds:

Lemma 3.2.1. *Let η be a smooth, closed, real $(1,1)$-form on the unit disc in $\mathbb{C}^m$. Then there exists a smooth real function ϕ on the unit disc such that $\eta = \mathrm{d}\mathrm{d}^c\phi$.*

Definition 3.2.4 (Kähler Potential)**.** Let M be a complex manifold with Kähler metric g and Kähler form ω. Then locally we may write $\omega = \mathrm{d}\mathrm{d}^c\phi$ for some real function ϕ. Such a function ϕ is called a *Kähler potential*.

Example 3.2.1 (Complex Euclidean Space)**.** Let us consider the space $M := \mathbb{C}^m \cong \mathbb{R}^{2m}$ with coordinates $z^\mu = x^\mu + \mathrm{i}y^\mu$ for $\mu = 1, \ldots, m$ and the euclidean metric d on $\mathbb{R}^{2m}$,

$$\begin{aligned} &d\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = d\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right) = \delta_{\mu\nu}, \\ &d\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu}\right) = 0, \end{aligned} \tag{3.35}$$

for $\mu, \nu = 1, \ldots, m$. We choose the complex structure

$$J\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial y^\mu}, \qquad J\left(\frac{\partial}{\partial y^\mu}\right) = -\frac{\partial}{\partial x^\mu}, \qquad \mu = 1, \ldots, m. \tag{3.36}$$

Then d is a Hermitian metric and we have

$$\begin{aligned} &d\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) = d\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) = 0, \\ &d\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right) = \frac{1}{2}\delta_{\mu\nu}. \end{aligned} \tag{3.37}$$

The Kähler form is given by

$$\omega := \frac{\mathrm{i}}{2}\sum_{\mu=1}^m \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\mu = \sum_{\mu=1}^m \mathrm{d}x^\mu \wedge \mathrm{d}y^\mu. \tag{3.38}$$

Then $\mathrm{d}\omega = 0$ and the Euclidean metric d of $\mathbb{R}^{2m}$ is a Kähler metric of $\mathbb{C}^m$. The Kähler potential is

$$\phi(z, \bar{z}) = \frac{1}{2}\sum_{\mu=1}^m z^\mu \bar{z}^\mu. \tag{3.39}$$

The Kähler manifold $\mathbb{C}^m$ is called the *complex Euclidean space.*

Example 3.2.2 (Complex Projective Space and Fubini-Study metric)**.** $\mathbb{CP}^m$ carries a natural Kähler metric, which may be defined as follows. First we have a natural projection

$$\pi : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{CP}^m, \qquad (z_0, \ldots, z_m) \mapsto [z_0, \ldots, z_m], \tag{3.40}$$

where $[z_0, \ldots, z_m]$ denote the homogeneous coordinates of a point z,

$$[z_0, \ldots, z_m] := \left\{x \in \mathbb{C}^{m+1} : (x_0, \ldots x_m) = \lambda(z_0, \ldots, z_m),\ \lambda \in \mathbb{C} \setminus \{0\}\right\}. \tag{3.41}$$

We then define a real function $u : \mathbb{C}^{m+1} \setminus \{0\} \to (0, \infty)$ by

$$u(z_0, \ldots, z_m) := |z_0|^2 + \cdots + |z_m|^2, \tag{3.42}$$

and a $(1,1)$-form α on $\mathbb{C}^{m+1} \setminus \{0\}$ by $\alpha := \mathrm{dd}^c(\log u)$. Note that α is not the Kähler form of any Kähler metric on $\mathbb{C}^{m+1} \setminus \{0\}$, because it is not positive. But there does exist a unique $(1,1)$-form ω on $\mathbb{CP}^m$, such that $\alpha = \pi^*(\omega)$. The Kähler metric g on $\mathbb{CP}^m$ with Kähler form ω is the *Fubini-Study metric*.

To make this more explicit, we use the inhomogeneous coordinates introduced in example 3.1.1. With these coordinates we define a positive-definite function in a chart U_a by

$$u_a : U_a \to \mathbb{R}, \qquad u_a(z_0, \ldots, z_m) := \sum_{\mu=0}^{m} \left| \frac{z^\mu}{z^a} \right|^2 = \frac{u(z_0, \ldots, z_m)}{|z_a|^2} - 1. \tag{3.43}$$

Given two charts U_a and U_b, the functions u_a and u_b are related on $z \in U_a \cap U_b$ by

$$u_a(z) = \left| \frac{z^b}{z^a} \right|^2 u_b(z). \tag{3.44}$$

Then it follows that

$$\ln u_a = \ln u_b + \ln \frac{z^b}{z^a} + \overline{\ln \frac{z^b}{z^a}}. \tag{3.45}$$

Since z^b/z^a is a holomorphic function, it follows that

$$\bar\partial \ln(z^b/z^a) = 0 \qquad \text{and} \qquad \partial\, \overline{\ln(z^b/z^a)} = 0. \tag{3.46}$$

This means

$$\partial\bar\partial \ln u_a = \partial\bar\partial \ln u_b, \tag{3.47}$$

and we can (locally) define a closed 2-form ω by (cf. (3.29))

$$\omega := 2\mathrm{i}\partial\bar\partial \ln u_a = \mathrm{dd}^c \ln u_a. \tag{3.48}$$

Now there exists a hermitian metric whose Kähler form is ω. We define

$$g \in \Gamma(T^*\mathbb{CP}^m \otimes T^*\mathbb{CP}^m), \qquad g(X,Y) := \omega(X, JY), \tag{3.49}$$

where J is the complex structure, which acts on (anti-)holomorphic vector fields in a chart by multiplication with $\mp\mathrm{i}$, cf. remark 3.1.2. Then g is indeed hermitian,

$$g(JX, JY) = -\omega(JX, Y) = \omega(Y, JX) = g(X, Y), \tag{3.50}$$

and one can show that g is positive definite, so that it is a metric. This metric is called the *Fubini-Study metric*, with a Kähler potential locally given by the functions u_a.

Definition 3.2.5 (Nearly Kähler Manifold)**.** An almost Hermitian manifold (M, J) is said to be *nearly Kähler*, if the following identity holds:

$$\nabla_X(J)Y = -\nabla_Y(J)X, \qquad \forall X, Y \in \mathfrak{X}(M). \tag{3.51}$$

Remark 3.2.2. An almost Hermitian manifold (M, J) is Kähler, if and only if

$$\nabla_X(J)Y = 0, \qquad \forall X, Y \in \mathfrak{X}(M), \tag{3.52}$$

see also proposition 3.2.1.

3.3 Curvature of Manifolds with Kähler Metrics

Let M be a $2m$-manifold with Kähler metric g. Using the notation introduced in remark 3.1.4, one can show that the components of the Riemannian curvature tensor of g satisfy

$$R^a{}_{bcd} = R^\alpha{}_{\beta\gamma\bar{\delta}} + R^\alpha{}_{\beta\bar{\gamma}\delta} + R^{\bar{\alpha}}{}_{\bar{\beta}\gamma\bar{\delta}} + R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}. \tag{3.53}$$

Now a general tensor T has 16 components in its complex decomposition (cf. remark 3.1.4). But equation (3.53) says that 12 out of the 16 components vanish, leaving 4 components. Using symmetries of the Riemannian curvature together with complex conjugation, we may identify $R^\alpha{}_{\beta\bar{\gamma}\delta}$ with $R^\alpha{}_{\beta\gamma\bar{\delta}}$, and identify both $R^{\bar{\alpha}}{}_{\bar{\beta}\gamma\bar{\delta}}$ and $R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}$ with the complex conjugate of $R^\alpha{}_{\beta\gamma\bar{\delta}}$. Thus the Kähler curvature is determined by the single component $R^\alpha{}_{\beta\gamma\bar{\delta}}$.

For the Ricci curvature $\mathrm{Ric} \in \Gamma(T^*M \otimes T^*M)$ with components $\mathrm{Ric}_{bd} := \sum_{a=1}^m R^a{}_{bad}$ we then get

$$\mathrm{Ric}_{bd} = \sum_{\alpha=1}^m R^\alpha{}_{\beta\alpha\bar{\delta}} + \sum_{\bar{\alpha}=1}^m R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\alpha}\delta}. \tag{3.54}$$

Definition 3.3.1 (Ricci Form)**.** We define the *Ricci form* by

$$\rho(X, Y) := \mathrm{Ric}(JX, Y), \qquad \forall X, Y \in \mathfrak{X}(M). \tag{3.55}$$

Remark 3.3.1. (i) The Ricci form ρ is a closed 2-form.

(ii) It can be shown that the components of the Ricci curvature are given by

$$\mathrm{Ric}_{\alpha\bar{\beta}} = -\partial_\alpha \bar{\partial}_{\bar{\beta}} \left[\log \det(g)\right] \tag{3.56}$$

Theorem 3.3.1. *For a Kähler manifold M of complex dimension m, the restricted linear holonomy group* Hol^0 *is contained in $SU(m)$ if and only if the Ricci tensor vanishes identically.*

Chapter 4

Manifolds with Special Holonomy

We now take a closer look at manifolds which have a holonomy group G, with G one of $Spin(7)$, G_2 and $SU(m)$. The case of a manifold M with holonomy $SU(3)$ is especially interesting, as one can construct manifolds $\mathbb{R} \times M$ with holonomy group G_2 from them. The references for this chapter are [2, 41, 51], and for 4.3 especially [14, 15].

4.1 The Groups $Spin(7)$ and G_2

Definition 4.1.1. The n-dimensional *spin group* $Spin(n)$ is defined as the double cover of the special orthogonal group in n dimensions, such that there exists a short exact sequence[1]

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1, \qquad \text{i.e.} \qquad \frac{Spin(n)}{\mathbb{Z}_2} \cong SO(n). \tag{4.1}$$

If we restrict ourselves to 7 dimensions, we also may take the following theorem as a definition of $Spin(7)$.

Theorem 4.1.1. *Consider $\mathbb{R}^8$ with coordinates $(x_1, \ldots, x_8)$. As a shorthand notation, we write $\mathrm{d}\mathbf{x}^{ijkl} := \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^l$. We define the 4-form*

$$\begin{aligned} \Omega_0 := {} & \mathrm{d}\mathbf{x}^{1234} + \mathrm{d}\mathbf{x}^{1256} + \mathrm{d}\mathbf{x}^{1278} + \mathrm{d}\mathbf{x}^{1357} - \mathrm{d}\mathbf{x}^{1368} \\ & - \mathrm{d}\mathbf{x}^{1458} - \mathrm{d}\mathbf{x}^{1467} - \mathrm{d}\mathbf{x}^{2358} - \mathrm{d}\mathbf{x}^{2367} - \mathrm{d}\mathbf{x}^{2457} \\ & + \mathrm{d}\mathbf{x}^{2468} + \mathrm{d}\mathbf{x}^{3456} + \mathrm{d}\mathbf{x}^{3478} + \mathrm{d}\mathbf{x}^{5678}. \end{aligned} \tag{4.2}$$

Then the group $Spin(7) \subset GL(8, \mathbb{R})$ is given by the elements $g \in GL(8, \mathbb{R})$ which leave Ω_0 invariant,

$$Spin(7) = \{g \in GL(8, \mathbb{R}) : g^* \Omega_0 = \Omega_0\}. \tag{4.3}$$

Remark 4.1.1. (i) From the first definition one may see that the dimension of $Spin(n)$ and $SO(n)$ are the same,

$$\dim Spin(n) = \dim SO(n) = \frac{n(n-1)}{2}, \tag{4.4}$$

and that their Lie algebras are isomorphic, especially $\mathfrak{spin}(7) \cong \mathfrak{so}(7)$.

[1] A sequence $G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} G_n$ is called an *exact sequence*, if $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$. A *short exact sequence* is an exact sequence of the form $1 \to A \to B \to C \to 1$. A is some subobject of B, so that $C \cong B/A$.

(ii) The group $Spin(7)$ is a compact, connected, simply connected, semisimple, 21-dimensional Lie group.

(iii) The 4-form Ω_0 in (4.2) is self-dual, i.e. $\Omega_0 = *\Omega_0$, where $*$ denotes the Hodge star operator.

(iv) Further the group $Spin(7)$ preserves the Euclidean metric $g_0 := (\mathrm{d}x^1)^2+\cdots+(\mathrm{d}x^8)^2$ and the orientation of $\mathbb{R}^8$.

Our aim now is to construct the exceptional Lie group G_2 as a subgroup of $Spin(7)$.

Definition 4.1.2. Let e be a unit vector in $\mathbb{R}^8$, i.e. $e \in S^7$. We define the group

$$G_2 := \{g \in Spin(7) : ge = e\}, \tag{4.5}$$

i.e. G_2 is the stabilizer subgroup of e in $Spin(7)$.

Remark 4.1.2. (i) The above definition especially means that we may regard G_2 as a subgroup of $GL(7,\mathbb{R})$.

(ii) One can show[2] that $Spin(7)$ acts transivitely on S^7, especially $S^7 = Spin(7)/G_2$ (cf. chapter 2.1). We then calculate

$$\dim G_2 = \dim Spin(7) - \dim S^7 = 21 - 7 = 14. \tag{4.6}$$

This may be compared with the more heuristic argumentation in remark A.2.4.

Another way to define G_2 is given in the following theorem.

Theorem 4.1.2. *Consider $\mathbb{R}^7$ with coordinates $(x^1,\ldots,x^7)$ and let $\mathrm{d}\mathbf{x}^{ijk} := \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$. We define the 3-form*

$$\phi_0 := \mathrm{d}\mathbf{x}^{123} + \mathrm{d}\mathbf{x}^{145} + \mathrm{d}\mathbf{x}^{167} + \mathrm{d}\mathbf{x}^{246} - \mathrm{d}\mathbf{x}^{257} - \mathrm{d}\mathbf{x}^{347} - \mathrm{d}\mathbf{x}^{356}. \tag{4.7}$$

Then G_2 is isomorphic to the subgroup of $GL(7,\mathbb{R})$ that leaves ϕ_0 invariant,

$$G_2 = \{g \in GL(7,\mathbb{R}) : g^*\phi_0 = \phi_0\}. \tag{4.8}$$

Remark 4.1.3. (i) The group G_2 is a compact, connected, simply-connected, semisimple and 14-dimensional Lie group, which also fixes the 4-form

$$*\phi_0 = \mathrm{d}\mathbf{x}^{4567} + \mathrm{d}\mathbf{x}^{2367} + \mathrm{d}\mathbf{x}^{2345} + \mathrm{d}\mathbf{x}^{1357} - \mathrm{d}\mathbf{x}^{1346} - \mathrm{d}\mathbf{x}^{1256} - \mathrm{d}\mathbf{x}^{1247}, \tag{4.9}$$

the metric $g_0 := (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^7)^2$ and the orientation of $\mathbb{R}^7$.

(ii) The group of automorphisms of the imaginary octonions introduced in appendix A.2 is isomorphic to G_2. As the notation in remark A.2.4 suggests, the Lie algebra of G_2 is isomorphic to the derivation algebra of the octonions, $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$.

[2] As a reference, see e.g. A. Gray, P. Green, *Sphere transitive structures and the triality automorphism*, Pacific journal of mathematics, **34** (1970), 83–96.

To give a quick overview of the relations between some of the groups introduced in appendix B and the groups $Spin(7)$ and G_2, we state the following result.

Theorem 4.1.3. *The only connected Lie subgroups of $Spin(7)$ which can be the holonomy group of a Riemannian metric on an 8-manifold are:*

(i) $\{1\}$,

(ii) $SU(2)$, *acting on* $\mathbb{R}^8 \cong \mathbb{R}^4 \oplus \mathbb{C}^2$ *trivially on* $\mathbb{R}^4$ *and as usual on* $\mathbb{C}^2$,

(iii) $SU(2) \times SU(2)$, *acting on* $\mathbb{R}^8 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ *in the obvious way,*

(iv) $SU(3)$, *acting on* $\mathbb{R}^8 \cong \mathbb{R}^2 \oplus \mathbb{C}^3$ *trivially on* $\mathbb{R}^2$ *and as usual on* $\mathbb{C}^3$,

(v) G_2, *acting on* $\mathbb{R}^8 \cong \mathbb{R} \oplus \mathbb{R}^7$ *trivially on* $\mathbb{R}$ *and as usual on* $\mathbb{R}^7$,

(vi) $Sp(2)$, *(vii)* $SU(4)$, *and (viii)* $Spin(7)$,

each of the last three acting on $\mathbb{R}^8$ *as usual.*

The inclusion "$\longrightarrow$" between the groups is shown below.

$$\begin{array}{ccccccc}
SU(2) & = & SU(2) & \longrightarrow & SU(3) & \longrightarrow & G_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
SU(2)\times SU(2) & \longrightarrow & Sp(2) & \longrightarrow & SU(4) & \longrightarrow & Spin(7)
\end{array}$$

4.2 7-Dimensional Manifolds with G_2-Structure

Before we define what we mean by a G_2-manifold, we need to introduce the notion of positivity of forms.

Definition 4.2.1. Let M be an oriented 7-manifold. For each $p \in M$, define P^3_pM to be the subset of 3-forms $\phi \in \Omega^3_p(M)$ for which there exists an oriented isomorphism between T_pM and $\mathbb{R}^7$, identifying ϕ and the 3-form ϕ_0 from theorem 4.1.2. Then P^3_pM is isomorphic to[3] $GL_+(7,\mathbb{R})/G_2$, since ϕ_0 has symmetry group G_2.

Now $\dim GL_+(7,\mathbb{R}) = 49$ and $\dim G_2 = 14$, so $GL_+(7,\mathbb{R})/G_2$ has dimension $49-14 = 35$. But $\Omega^3_p(M)$ has also dimension $\binom{7}{3} = 35$, so $P^3_p(M)$ is an open subset of $\Omega^3_p(M)$.

Definition 4.2.2. Let P^3M be the bundle over M with fibre P^3_pM at each $p \in M$. Then P^3M is an open subbundle[4] of $\Omega^3(M)$ with fibre $GL_+(7,\mathbb{R})/G_2$. We say that a 3-form ϕ on M is *positive*, if $\phi_{|p} \in P^3_pM$ for all $p \in M$.

[3] $GL_+(n,\mathbb{R})$ denotes the identity component of the group $GL(n,\mathbb{R})$.
[4] Because of the dimensions of the spaces, this is not a vector subbundle.

Definition 4.2.3 (G_2-Manifold)**.** Let M be an oriented 7-manifold, ϕ a positive 3-form on M, and g the associated metric. Let ∇ be the Levi-Civita connection of g. Since the pair (ϕ, g) defines a unique G_2-structure on M, we refer to (ϕ, g) as a G_2-structure. We call $\nabla\phi$ the *torsion* of (ϕ, g). If $\nabla\phi = 0$, then (ϕ, g) is called *torsion-free.* We define a *G_2-manifold* to be a triple (M, ϕ, g), where M is a 7-manifold and (ϕ, g) a torsion-free G_2-structure on M.

Theorem 4.2.1. *A 7-dimensional, oriented Riemannian manifold (M, g) is a G_2-manifold if and only if its holonomy group is contained in G_2.*

Proposition 4.2.1. *Let M be a 7-manifold and (ϕ, g) a G_2-structure on M. Then the following are equivalent:*

(i) (ϕ, g) *is torsion-free,*

(ii) $\mathrm{Hol}(g) \subseteq G_2$, *and* ϕ *is the induced 3-form,*

(iii) $\nabla\phi = 0$ *on* M, *where* ∇ *is the Levi-Civita connection of* g, *and*

(iv) $\mathrm{d}\phi = \mathrm{d}^*\phi = 0$ *on* M.

Proposition 4.2.2. *Let (M, ϕ, g) be a compact G_2-manifold. Then $\mathrm{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite.*

To close the chapter on G_2-manifolds, we now give a construction method for these spaces.

Theorem 4.2.2. *Suppose (M, g_M) is a Riemannian 6-manifold with holonomy $SU(3)$. Then M admits a complex structure J, a Kähler form ω and a holomorphic volume form θ with $\mathrm{d}\omega = \mathrm{d}\theta = 0$.*

Let $\mathbb{R}$ have coordinate x. Define a metric g and a 3-form ϕ on $\mathbb{R} \times M$ by

$$g := \mathrm{d}x^2 + g_M \quad \textit{and} \quad \phi := \mathrm{d}x \wedge \omega + \mathrm{Re}\,\theta. \tag{4.10}$$

Then (ϕ, g) is a torsion-free G_2-structure on $\mathbb{R} \times M$, and

$$*\phi = \frac{1}{2}\omega \wedge \omega - \mathrm{d}x \wedge \mathrm{Im}\,\theta. \tag{4.11}$$

4.3 Nearly Kähler Manifolds and 3-Symmetric Spaces

Consider a Riemannian manifold (M, g) of dimension $n = 2m$. Then a $SU(m)$-structure is a reduction of the structure group to $SU(m)$. A special case is given in the following definition.

Definition 4.3.1. A *Calabi-Yau manifold* is a compact Kähler manifold (M, J, g) of dimension $m \geq 2$, with $\mathrm{Hol}(g) = SU(m)$.

We now investigate the case of nearly Kähler manifolds, as introduced before.

Definition 4.3.2. Given a Riemannian manifold (M, g), its *Riemannian cone* is a product $M \times \mathbb{R}^{>0}$ of M with the half-line $\mathbb{R}^{>0}$, which is equipped with the *cone metric* $\mathrm{d}s^2 = t^2 g + \mathrm{d}t^2$, where t denotes the parameter in $\mathbb{R}^{>0}$.

Proposition 4.3.1. *A Riemannian manifold* (M, g) *with* $\dim M = 6$ *carries a nearly Kähler structure if and only if its cone* $(M \times \mathbb{R}^{>0}, t^2 g + \mathrm{d}t^2)$ *has a holonomy contained in* G_2*.*

Nearly Kähler manifolds of compact type may be classified in terms of 3-symmetric spaces, which are defined in the following.

Definition 4.3.3. A Riemannian manifold (M, g) is called a 3-*symmetric space*, if it admits a family of isometries $\{\theta_p\}_{p\in M}$ of (M, g) satisfying

(i) $\theta_p^3 = \mathrm{id}$,

(ii) p is an isolated fixed point of θ_p,

(iii) the tensor field g defined by $\Theta := (\theta_p)_{*p}$ is of class $\mathscr{C}^\infty$,

(iv) $\theta_{p*} \circ J = J \circ \theta_{p*}$,

where J is the *canonical almost complex structure* of the family $\{\theta_p\}_{p\in M}$ given by $J = \frac{1}{2\sqrt{3}}(\Theta + \mathrm{id})$.

Remark 4.3.1. Riemannian 3-symmetric spaces are characterised by a triple $(G/K, \sigma, \langle\cdot,\cdot\rangle)$, satisfying the following conditions:

(i) G is a connected Lie group and σ is an automorphism of G of order 3.

(ii) K is a closed subgroup of G, such that $G_0^\sigma \subseteq K \subseteq G^\sigma$, where G^σ is the stabilizer subgroup of σ and G_0^σ the identity component.

(iii) $\langle\cdot,\cdot\rangle$ is an $\mathrm{Ad}\,(K)$- and σ-invariant inner product on the vector space $\mathfrak{m} = (\mathfrak{m}^+ \oplus \mathfrak{m}^-) \cap \mathfrak{g}$, where $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are the eigenspaces of σ on the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$ corresponding to the eigenvalues $\varepsilon = \mathrm{e}^{2\pi\mathrm{i}/3}$ and $\varepsilon^2 = \mathrm{e}^{-2\pi\mathrm{i}/3}$.

Theorem 4.3.1. *A simply connected, compact Riemannian* 3*-symmetric space with strictly positive sectional curvature is isometric to one of the following spaces, equipped with a suitable invariant metric:*

(i) $\mathbb{CP}^n = SU(n+1)/S(U(1) \times U(n))$,

(ii) $\mathbb{F}^6 = SU(3)/U(1) \times U(1)$,

(iii) $\mathbb{CP}^n = Sp(m)/Sp(m-1) \times U(1)$, $n = 2m - 1$,

(iv) $S^6 = G_2/SU(3)$.

Lemma 4.3.1. *Any simply connected, irreducible non-Kähler homogeneous nearly Kähler manifold* (M, J, g) *is a compact* 3*-symmetric space.*

Proposition 4.3.2. *Any non-Kähler homogeneous nearly Kähler manifold with strictly positive sectional curvature is holomorphically isometric to one of the following* 3*-symmetric spaces with respect to the canonical complex structure:*

(i) $\mathbb{CP}^n = Sp(m)/Sp(m-1) \times U(1)$, $n = 2m - 1$,

(ii) $S^6 = G_2/SU(3)$.

Chapter 5

Yang-Mills Theory

We give a brief introduction to Yang-Mills theory on principal bundles, as it can be found in [35, 38]. To simplify the notation, we are going to use the *Einstein summation convention* from now on, i.e. repeated indices are meant to be summed over from 1 to the dimension n of the space. To express the local curvature form $\mathcal{F}$ in its components we will for example write

$$\mathcal{F} = \frac{1}{2}\mathcal{F}_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b := \frac{1}{2}\sum_{a,b=1}^{n} \mathcal{F}_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b. \tag{5.1}$$

5.1 Yang-Mills Theory on Principal Bundles

Consider a principal bundle $P(M,G)$ over a manifold M with structure group G. In this context the Lie group G is also often called the *gauge group*. Let $\mathfrak{g}$ denote the Lie algebra of G.

Definition 5.1.1 (Functional)**.** We call a continuous mapping S between two normed spaces V, W an *operator*. If W is a field, we call S a *functional*.

Definition 5.1.2 (Action)**.** Let (M,g) and (N,h) be Riemannian manifolds. M is called *base manifold*, N the *target manifold*. The functional

$$\mathcal{S} : \Gamma(M,N) \to \mathbb{R}, \qquad \mathcal{S}[\phi] := \int_M \mathcal{L}(\phi, D\phi, D^2\phi, \ldots, D^k\phi, x), \tag{5.2}$$

where $\Gamma(M,N)$ denotes the set of smooth sections from M to N and $D^k\phi$ is the k-th covariant derivative of ϕ, is called the *action* or *action functional*. The integrand $\mathcal{L} : N \times TN \times T^2N \times \cdots \times T^kN \times M \to \mathbb{R}$ is called the *Lagrangian density*[1].

Remark 5.1.1. For our purposes, the Lagrangian density is considered to depend only on the functions ϕ and their first derivatives $D\phi$.

Definition 5.1.3 (Gauge Field Lagrangian)**.** Let $P(M,G)$ be a principal bundle and $U \subset M$ a chart with local curvature form $\mathcal{F}$. Then the *(pure) Yang-Mills Lagrangian* or *gauge field Lagrangian* on U is given by

$$\mathcal{L}_{\mathrm{YM}} := \mathrm{Tr}(\mathcal{F} \wedge *\mathcal{F}). \tag{5.3}$$

[1] More generally, a Lagrangian density is defined on a *jet bundle* [35], but this definition should suffice for our purposes.

Remark 5.1.2. In the previous definition we suppressed the dependencies on the functions ϕ to keep the notation a bit more handy. In general we have that the Lagrangian $\mathcal{L} = \mathcal{L}[\phi]$ and the curvature form $\mathcal{F} = \mathcal{F}[\phi]$ depend on the functions ϕ.

Remark 5.1.3. We consider the gauge field Lagrangian (5.3). Using equation (5.1), the Lagrangian may be slightly rewritten as

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{4}\mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})\mathrm{vol}_n, \tag{5.4}$$

where $\mathrm{vol}_n := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ denotes the n-dimensional volume element.

Remark 5.1.4 (Gauge Transformations)**.** Let $P(M, G)$ be a principal bundle and let $\pi : P \to M$ denote its projection.

(i) We call the choice of a local section $s \in \Gamma(U, \pi^{-1}(U))$ on a chart $U \subset M$ the choice of a *gauge* (cf. remark C.3.3).

(ii) Consider two charts $U, V \subset M$ with $U \cap V \neq 0$ and local sections s_1, s_2 on U resp. V. Then on $U \cap V$ they are related via

$$s_2 = s_1 g, \tag{5.5}$$

with a transition function g. Given local connection forms $\mathcal{A}$ on U, $\mathcal{A}'$ on V, these two are related on $U \cap V$ via

$$\mathcal{A}_\mu = g^{-1}\mathcal{A}'_\mu g + g^{-1}\partial_\mu g. \tag{5.6}$$

We call this relation a *gauge transformation* (cf. remark C.3.4). If g has no dependence on the coordinate $x \in U \cap V$, we call this transformation *global*, otherwise *local*.

(iii) Let $P(M, G)$ be a principal bundle, and $U, V \subset M$ two charts with $U \cap V \neq 0$. Then by definition C.3.9 we have

$$\mathcal{F}_U = s_U^*\Omega, \qquad \mathcal{F}_V = s_V^*\Omega \tag{5.7}$$

on U resp. V. On the overlap of the charts, $\mathcal{F}_U$ and $\mathcal{F}_V$ have to satisfy

$$\mathcal{F}_V = h_{UV}^{-1}\mathcal{F}_U h_{UV}, \tag{5.8}$$

where h_{UV} is the transition function (cf. equation (C.23)). Note that (5.3) is invariant under such transformations, so that we can write down the action $\mathcal{S}_{\mathrm{YM}}$ of the gauge field Lagrangian on M,

$$\mathcal{S}_{\mathrm{YM}} = \int_M \mathrm{Tr}(\mathcal{F} \wedge *\mathcal{F}), \tag{5.9}$$

which then is invariant under the action of the gauge group G.

The equations of motion for the pure Yang-Mills theory are given by

$$D * \mathcal{F} = 0, \tag{5.10}$$

which is obtained by extremizing the action (5.9). In components, they read

$$D_\mu \mathcal{F}^{\mu\nu} = \partial_\mu \mathcal{F}^{\mu\nu} + [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = 0, \tag{5.11}$$

where D denotes the exterior covariant derivative.

Remark 5.1.5 (Yang-Mills Theory with Interacting Massive Fields)**.** Let $P(M,G)$ be a principal bundle and $F := P \times_G X$ an associated bundle, where X is a Riemannian manifold on which G acts by isometries. Let further $\phi \in \Gamma(F, M)$ be a section in the associated bundle (ϕ is called a *field*), $U \subset M$ a chart, $s \in \Gamma(U, \pi^{-1}(U))$ a local section, and $\varphi := s^*\phi$. We define $\mathcal{L}_0$, which is called the *free Lagrangian* (i.e. without the interaction terms) on U, by

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}(\langle \mathcal{D}_\mu \varphi, \mathcal{D}^\mu \varphi \rangle - m^2 \langle \varphi, \varphi \rangle) \\ &:= \frac{1}{2}(s^*\langle D_\mu \phi, D^\mu \phi \rangle - m^2 s^* \langle \phi, \phi \rangle), \end{aligned} \tag{5.12}$$

where $\langle \cdot, \cdot \rangle : U \times U \to \mathbb{R}$ denotes the (induced) inner product on the chart U and $\mathcal{D}$ is defined on U by

$$\mathcal{D}(\varphi) = \mathcal{D}(s^*\phi) := s^*(D\phi). \tag{5.13}$$

The full Lagrangian now is given by

$$\begin{aligned} \mathcal{L} &:= \mathcal{L}_0 + \mathcal{L}_I \\ &= \frac{1}{2}\langle \mathcal{D}_\mu \varphi, \mathcal{D}^\mu \varphi \rangle - \frac{1}{2} m^2 \langle \varphi, \varphi \rangle + \mathrm{Tr}(\mathcal{F} \wedge *\mathcal{F}). \end{aligned} \tag{5.14}$$

It is often convenient to view ϕ as an equivariant map $\phi : P \to X$. Since ϕ is a section in the associated bundle F (cf. definition C.1.2), the action of the gauge group G is given by

$$g \circ \phi(p) = g^{-1}\phi(gp), \quad g \in G,\ p \in P. \tag{5.15}$$

With remark C.3.7, we may write $\mathcal{D}\varphi = \mathrm{d}\varphi + \mathcal{A}\varphi$. Further, using $\mathrm{d}g^{-1} = -g^{-1}(\mathrm{d}g)g^{-1}$, the action of the gauge group on $\mathcal{D}\varphi$ is given by

$$\begin{aligned} g \circ (\mathcal{D}\varphi) &= g \circ (\mathrm{d}\varphi) + g \circ (\mathcal{A}\varphi) \\ &= (\mathrm{d}g^{-1})\varphi + g^{-1}(\mathrm{d}\varphi) + (g^{-1}\mathrm{d}g + g^{-1}\mathcal{A}g)(g^{-1}\varphi) \\ &= g^{-1}(\mathrm{d}\varphi + \mathcal{A}\varphi) + (\mathrm{d}g^{-1})\varphi + g^{-1}(\mathrm{d}g)g^{-1}\varphi \\ &= g^{-1}(\mathcal{D}\varphi). \end{aligned} \tag{5.16}$$

Since G acts by isometries, $\mathcal{L}$ is invariant under G. In particular $\mathcal{L}$ is invariant under the change of a local section, so that we can write down the action on M,

$$\mathcal{S} = \int_M \left[\frac{1}{2}\langle \mathcal{D}_\mu \varphi, \mathcal{D}^\mu \varphi \rangle - \frac{1}{2} m^2 \langle \varphi, \varphi \rangle + \mathrm{Tr}(\mathcal{F} \wedge *\mathcal{F}) \right]. \tag{5.17}$$

5.2 Anti-Self-Duality and Yang-Mills Equations with Torsion

We now want to generalize the equations (5.10) a bit. Let Σ be a $(d-4)$-form on a d-dimensional Riemannian manifold M. Consider the complex vector bundle $\mathcal{E}$ over M endowed with a connection $\mathcal{A}$. The Σ-*anti-self-dual gauge equations* are defined (see also [22]) as the first-order equations

$$*\mathcal{F} = -\Sigma \wedge \mathcal{F}. \tag{5.18}$$

Here, $*$ denotes the Hodge star operator on M.

Differentiating (5.18), we obtain the *Yang-Mills equations with torsion*,

$$\mathrm{d} * \mathcal{F} + \mathcal{A} \wedge *\mathcal{F} - *\mathcal{F} \wedge \mathcal{A} + *\mathcal{H} \wedge \mathcal{F} = 0, \tag{5.19}$$

where the torsion three-form $\mathcal{H}$ is defined by the formula

$$*\mathcal{H} := \mathrm{d}\Sigma, \tag{5.20}$$

so that

$$\mathcal{H} = (-1)^{3(d-3)} * \mathrm{d}\Sigma. \tag{5.21}$$

Note that if Σ is closed, $\mathcal{H} = 0$ and (5.19) reduces to the standard Yang-Mills equations (5.10).

Remark 5.2.1. The Yang-Mills equations with torsion (5.19) are the equations of motion for the action

$$\begin{aligned} \mathcal{S} &= \int_M \mathrm{Tr}\left[\mathcal{F} \wedge *\mathcal{F} + (-1)^{d-3}\Sigma \wedge \mathcal{F} \wedge \mathcal{F}\right] \\ &= \int_M \mathrm{Tr}\left[\mathcal{F} \wedge *\mathcal{F} + *\mathcal{H} \wedge \left(\mathrm{d}\mathcal{A} \wedge \mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right)\right] \\ &\qquad - \int_M \mathrm{d}\left[\Sigma \wedge \mathrm{Tr}\left(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right)\right], \end{aligned} \tag{5.22}$$

where the last term is called a *topological term.*

Chapter 6

Yang-Mills Equations on Homogeneous G_2-Manifolds

In this chapter, we take the Yang-Mills equations and analyze them on spaces of the form $\mathbb{R} \times G/H$, which have structure group G_2. Results from this chapter can also be found in [10].

6.1 Yang-Mills Fields on $\mathbb{R} \times G/H$

6.1.1 Coset Spaces. We consider the Yang-Mills equations on coset spaces of the form $\mathcal{M} := \mathbb{R} \times G/H$, with G/H either $SU(3)/U(1) \times U(1)$ or $Sp(2)/Sp(1) \times U(1)$. These manifolds are reductive homogeneous spaces, cf. example 2.1.3. Further they are 3-symmetric nearly Kähler spaces (cf. theorem 4.3.1, lemma 4.3.1 and proposition 4.3.2) and the space $\mathcal{M}$ admits a G_2-structure.

On the space G we choose the generators $\{I_A\}$, $A = 1, \ldots, \dim G$ with structure constants $f_{AB}{}^C$, such that the commutation relation

$$[I_A, I_B] = f_{AB}{}^C I_C \tag{6.1}$$

holds. Further we use the metric induced by the Cartan-Killing form (cf. definition B.5.2), which in components is given by

$$g_{AB} = B_{AB} = f^C{}_{AD} f^D{}_{CB}. \tag{6.2}$$

We may choose the Lie algebra generators in a way that g_{AB} is normalized, i.e. it is given by the Kronecker symbol,

$$g_{AB} = \delta_{AB}. \tag{6.3}$$

The Lie algebra $\mathfrak{g}$ of G can be decomposed $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of the Lie algebra $\mathfrak{h}$ of H in $\mathfrak{g}$. The generators of G can be divided into two sets, $\{I_a\}$ and $\{I_i\}$, where the $\{I_i\}$ are the generators of H with $i, j, \ldots = \dim G - \dim H + 1, \ldots, \dim G$, and $\{I_a\}$ span the subspace $\mathfrak{m}$ of $\mathfrak{g}$ with $a, b = 1, \ldots, \dim G - \dim H$. For reductive homogeneous spaces we then have the following commutation relations, as noted in remark 2.1.2:

$$[I_i, I_j] = f_{ij}{}^k I_k, \qquad [I_i, I_a] = f_{ia}{}^b I_b, \qquad [I_a, I_b] = f_{ab}{}^i I_i + f_{ab}{}^c I_c. \tag{6.4}$$

The components of the metric on $\mathfrak{g}$ are given by

$$g_{ia} = 0, \tag{6.5}$$

$$g_{ij} = f_{ikl}f_j^{\ kl} + f_{ibc}f_j^{\ bc} = \delta_{ij}, \tag{6.6}$$

$$g_{ab} = f_{acd}f_b^{\ cd} + 2f_{aci}f_b^{\ ci} = \delta_{ab}. \tag{6.7}$$

6.1.2 Torsionful Spin Connection on G/H. The metric 6.7 lifts to a G-invariant metric on G/H. We now construct a local expression of the metric by introducing an orthonormal frame.

The basis elements of the Lie algebra $\mathfrak{g}$ can be represented by left-invariant vector fields $\hat{E}_A$ on the Lie group G (cf. definition B.1.10 and remark B.1.1), and the dual basis $\hat{e}^A$ then is a set of left-invariant one-forms.

The space G/H consists of left cosets gH and the natural projection $g \mapsto gH$ is denoted $\pi : G \to G/H$. Over a small contractible open subset U of G/H, one can choose a map $L : U \to G$ such that $\pi \circ L$ is the identity, i.e. L is a local section of the principal bundle $G \to G/H$. The pull-backs of $\hat{e}^A$ by L are denoted e^A. Among these, the e^a form an orthonormal frame for $T^*(G/H)$ over U, and for the remaining forms we can write $e^i = e^i_{\ a}e^a$ with real functions $e^i_{\ a}$. The dual frame for $T(G/H)$ will be denoted E_a. By the group action we can transport e^a and E_a from inside U to everywhere in G/H. The forms e^A obey the Maurer-Cartan equations (cf. theorem B.5.2 resp. remark B.5.1),

$$\mathrm{d}e^a = -f^a_{\ ib}e^i \wedge e^b - \frac{1}{2}f^a_{\ bc}e^b \wedge e^c, \tag{6.8}$$

$$\mathrm{d}e^i = -\frac{1}{2}f^i_{\ bc}e^b \wedge e^c - \frac{1}{2}f^i_{\ jk}e^j \wedge e^k. \tag{6.9}$$

The local expression for the G-invariant metric then is

$$g_{G/H} = \delta_{ab}e^a e^b. \tag{6.10}$$

A linear connection $\omega = (\omega^a_{\ b}) = (\omega^a_{\ bc}e^c)$ is a matrix of one-forms (see e.g. remark C.3.6). The connection is metric compatible, if $g_{ab}\omega^c_b$ is anti-symmetric, and its torsion is a vector of two-forms T^a determined by the structure equations

$$\mathrm{d}e^a + \omega^a_{\ b} \wedge e^b = T^a = \frac{1}{2}T^a_{\ bc}e^b \wedge e^c. \tag{6.11}$$

We use a non-vanishing torsion tensor on G/H, where we choose the components to be proportional to the structure constants,

$$T^a_{\ bc} = \kappa f^a_{\ bc}, \tag{6.12}$$

and $\kappa \in \mathbb{R}$ is an arbitrary parameter. The torsionful spin connection on G/H then becomes

$$\omega^a_{\ b} = \omega^a_{\ bc}e^c = (e^i_{\ c}f^a_{\ ib} + \frac{1}{2}(\kappa + 1)f^a_{\ bc})e^c, \tag{6.13}$$

with $e^i = e^i_{\ c}e^c$.

6.1.3 Yang-Mills Equations on $\mathbb{R} \times G/H$. Consider the space $\mathbb{R} \times G/H$ with a coordinate τ on $\mathbb{R}$, a one-form $e^0 := \mathrm{d}\tau$ and the Euclidean metric

$$g := (e^0)^2 + \delta_{ab} e^a e^b, \tag{6.14}$$

which means we can pull down all indices. For the torsion-full spin connection (6.13) on $\mathcal{M}$ we have additional (vanishing) components,

$$\omega^0{}_{0b} = \omega^a{}_{0b} = \omega^0{}_{cb} = 0. \tag{6.15}$$

We introduce a three-form $\mathcal{H}$ on $\mathcal{M}$ with

$$\mathcal{H} := \frac{1}{3!} T_{abc} e^a \wedge e^b \wedge e^c = \frac{1}{6} \kappa f_{abc} e^a \wedge e^b \wedge e^c, \tag{6.16}$$

so that

$$\mathcal{H}_{abc} = T_{abc} = \kappa f_{abc}. \tag{6.17}$$

Now consider the trivial principal bundle $P(\mathcal{M}, G) \cong (\mathbb{R} \times G/H) \times G$ over $\mathbb{R} \times G/H$ with the structure group G, the associated trivial complex vector bundle $\mathcal{E}$ over $\mathbb{R} \times G/H$ and a $\mathfrak{g}$-valued connection one-form $\mathcal{A}$ on $\mathcal{E}$ with curvature form $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. In the basis of one-forms $\{e^0, e^a\}$ on $\mathbb{R} \times G/H$, we have

$$\mathcal{A} = \mathcal{A}_0 e^0 + \mathcal{A}_a e^a, \qquad \mathcal{F} = \mathcal{F}_{0a} e^0 \wedge e^a + \frac{1}{2} \mathcal{F}_{ab} e^a \wedge e^b. \tag{6.18}$$

In the following we choose a “temporal” gauge $\mathcal{A}_0 \equiv \mathcal{A}_\tau := 0$.

For our choice of $\mathcal{H}$ and ω, the Yang-Mills equations with torsion (5.19) are now given by

$$E_a \mathcal{F}^{a0} + \omega^a{}_{ab} \mathcal{F}^{b0} + [\mathcal{A}_a, \mathcal{F}^{0a}] = 0, \tag{6.19}$$

$$E_0 \mathcal{F}^{0b} + E_a \mathcal{F}^{ab} + \omega^d{}_{da} \mathcal{F}^{ab} + \omega^b{}_{cd} \mathcal{F}^{cd} + [\mathcal{A}_a, \mathcal{F}^{0a}] = 0, \tag{6.20}$$

where $E_0 = \mathrm{d}/\mathrm{d}\tau$. These equations also follow from the action functional (5.22) for our choice of $\mathcal{H}$ and with the gauge $\mathcal{A}_0 = 0$.

6.1.4 G-invariant Gauge Fields. Let us take our complex vector bundle $\mathcal{E} \to \mathbb{R} \times G/H$ to be of rank $\dim G$ and carry the adjoint representation $\mathrm{Ad}\, G$ of the structure group G. Then the generators of G are realized as $\dim G \times \dim G$ unitary matrices

$$I_i = \left(I_{iA}{}^B\right) = \left(f_{iB}{}^A\right) = \left(f_{ik}{}^j\right) \oplus \left(f_{ib}{}^a\right) \quad \text{and} \quad I_a = \left(I_{aA}{}^B\right) = \left(f_{aA}{}^B\right). \tag{6.21}$$

According to (2.13) (resp. [49]), G-invariant connections on $\mathcal{E}$ are determined by linear maps $\Lambda : \mathfrak{m} \to \mathfrak{g}$ which commute with the adjoint action of H:

$$\Lambda(\mathrm{Ad}\,(h) Y) = \mathrm{Ad}\,(h) \Lambda(Y), \qquad \forall h \in H \quad \text{and} \quad Y \in \mathfrak{m}. \tag{6.22}$$

Such a linear map is represented by a matrix $({X_a}^B)$, appearing in

$$X_a := \Lambda(I_a) = {X_a}^B I_B = {X_a}^i I_i + {X_a}^b I_b. \tag{6.23}$$

For the cases we consider one can always choose ${X_a}^i = 0$. In local coordinates the connection is written

$$\mathcal{A} = e^i I_i + e^a X_a \quad \Leftrightarrow \quad \mathcal{A}_a = {e_a}^i I_i + X_a, \tag{6.24}$$

and its G-invariance imposes the condition

$$[I_i, X_a] = {f_{ia}}^b X_b \quad \Leftrightarrow \quad {X_a}^b {f_{bi}}^c = {f_{ia}}^b {X_b}^c. \tag{6.25}$$

The curvature $\mathcal{F}$ of the connection (6.24) reads

$$\begin{aligned}\mathcal{F} &= \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \\ &= \dot{X}_a e^0 \wedge e^a - \frac{1}{2}\left({f_{bc}}^i I_i + {f_{bc}}^i I_i + {f_{bc}}^a X_a - [X_b, X_c]\right) e^b \wedge e^c,\end{aligned} \tag{6.26}$$

which means for the components

$$\mathcal{F}_{0a} = \dot{X}_a, \qquad \mathcal{F}_{bc} = -({f_{bc}}^i I_i + {f_{bc}}^a X_a - [X_b, X_c]), \tag{6.27}$$

where the dot denotes a derivative with respect to τ.

6.1.5 Explicit Calculation of the Equations of Motion. Inserting (6.24) and (6.27) into (6.19), using $E_a\mathcal{F}^{a0} = 0$ and ${\omega^d}_{db}\mathcal{F}^{b0} = e^i_c {f^c}_{ib}\dot{X}^b$, we get

$$\begin{aligned}0 &= E_a\mathcal{F}^{a0} + {\omega^d}_{db}\mathcal{F}^{b0} + [e^i_a I_i + X_a, \dot{X}^a] \\ &= e^i_c {f^c}_{ib}\dot{X}^b + e^i_c {f_i}^c{}_b \dot{X}^b + [X_a, \dot{X}^a] \\ &= [X_a, \dot{X}^a].\end{aligned} \tag{6.28}$$

This is the Gauss-law constraint[1] following from the gauge fixing $\mathcal{A}_0 = 0$.

For the equation (6.20), we begin by calculating the different terms in the sum.

(i) We get $E_0\mathcal{F}^{0b} = \ddot{X}^b$ and $E_a\mathcal{F}^{ab} = 0$, since we do not have any coordinate-dependence in $\mathcal{F}$.

(ii) We now consider ${\omega^d}_{da}\mathcal{F}^{ab}$.

$${\omega^d}_{da}\mathcal{F}^{ab} = e^i_c\left(-{f^c}_{ia} f^{abj} I_j - {f^c}_{ia} f^{abd} X_d + {f^c}_{ia}[X^a, X^b]\right). \tag{6.29}$$

[1] This constraint corresponds to the Gauß law in electrodynamics, hence the name. It is obtained from a Lagrangian $\mathcal{L}$ by varying with respect to $\mathcal{A}_0$, $\pi_0 = \mathcal{D}_0\mathcal{L}$. It comes from the fact that $\mathcal{A}_0$ enters the Lagrangian without time derivatives and that its associated canonical momentum vanishes, $\pi_0 = 0$. See e.g. [39, chapter 15.4].

(iii) From the fourth term $\omega^b{}_{cd}\mathcal{F}^{cd}$ we get

$$\begin{aligned}\omega^b{}_{cd}\mathcal{F}^{cd} &= -e^i_c\left(f^b{}_{id}f^{cdj}I_j - f^b{}_{id}f^{cda}X_a + f^b{}_{id}[X^c, X^d]\right)\\ &\quad - \frac{1}{2}(\kappa+1)\left(f^b{}_{cd}f^{cdj}I_j + f^b{}_{cd}f^{cda}X_a + f^b{}_{cd}[X^c, X^d]\right).\end{aligned} \tag{6.30}$$

Now using the equations for the components of the metric (6.5)-(6.7), we can simplify this further to

$$\begin{aligned}\omega^b{}_{cd}\mathcal{F}^{cd} &= -e^i_c\left(f^b{}_{id}f^{cdj}I_j - f^b{}_{id}f^{cda}X_a + f^b{}_{id}[X^c, X^d]\right)\\ &\quad - \frac{1}{2}(\kappa+1)\left(X_b + f^b{}_{cd}[X^c, X^d]\right).\end{aligned} \tag{6.31}$$

(iv) The commutator $[\mathcal{A}_a, \mathcal{F}^{ab}]$ is calculated to be

$$\begin{aligned}[\mathcal{A}_a, \mathcal{F}^{ab}] &= -e^i_a\left(f^{abj}[I_i, I_j] + f^{abd}[I_i, X_d] - [I_i, [X^a, X^b]]\right)\\ &\quad - f^{abj}[X_a, I_j] - f^{abd}[X_a, X_d] + [X_a, [X^a, X^b]].\end{aligned} \tag{6.32}$$

Using the G-invariance condition (6.25) and the Jacobi identities,

$$\begin{aligned}[I_i, [X^a, X^b]] &= [X^a, [I_i, X^b]] - [X^b, [I_i, X^a]]\\ &= f_i{}^{bc}[X^a, X_c] - f_i{}^{ac}[X^b, X_c],\end{aligned} \tag{6.33}$$

we can rewrite equation (6.32) into

$$\begin{aligned}[\mathcal{A}_a, \mathcal{F}^{ab}] &= -e^i_a\left(f^{abj}f_{ij}{}^kI_k + f^{abd}f_{id}{}^cX_c - f_i{}^{bc}[X^a, X_c] - f_i{}^{ac}[X^b, X_c]\right)\\ &\quad + f^{abj}f_{ja}{}^cX_c - f^{abd}[X_a, X_d] + [X_a, [X^a, X^b]].\end{aligned} \tag{6.34}$$

Before now adding up the different parts of the equation (6.20), we make the following observations. First $f^A{}_{BC}f^{DBC} = \delta^{AD}$ together with $f_{aij} = 0$ implies $f^b{}_{cd}f^{cdj} = 0$. Using (6.7), we further get $f^b{}_{cd}f^{cda} = \frac{1}{3}\delta^{ba}$. So from (6.20) and the addition of (6.29), (6.30), (6.31) and (6.34) we obtain the lengthy expression

$$\begin{aligned}\ddot{X}^a &= e^i_c\left(f^c{}_{ib}f^{baj}I_j + f^c{}_{ib}f^{bad}X_d - f^c{}_{ib}[X^b, X^a]\right)\\ &\quad + e^i_c\left(f^a{}_{id}f^{cdj}I_j - f^a{}_{id}f^{cdb}X_b + f^a{}_{id}[X^c, X^d]\right)\\ &\quad + e^i_c\left(f^{caj}f_{ij}{}^kI_k + f^{cad}f_{id}{}^bX_b - f_i{}^{ab}[X^c, X_b] - f_i{}^{cb}[X^a, X_b]\right)\\ &\quad + \frac{1}{2}(\kappa+1)\left(\frac{1}{3}X_a + f^a{}_{cd}[X^c, X^d]\right)\\ &\quad + \frac{1}{2}(\kappa+1)\left(X_a + f^a{}_{cd}[X^c, X^d]\right)\\ &\quad - f^{baj}f_{jb}{}^cX_c + f^{bad}[X_b, X_d] - [X_b, [X^b, X^a]].\end{aligned} \tag{6.35}$$

Using the Jacobi identity, the terms proportional to ($e^i_c \times$ Lie-algebra generator) add up to zero. With the same argument, the terms proportional to ($e^i_c \times X^d$) also vanish. Now pulling down all the indices, we arrive at the equation

$$\ddot{X}_a = \left(\frac{1}{2}(\kappa+1)f_{acd}f_{bcd} - f_{acj}f_{bcj}\right)X_b - \frac{1}{2}(\kappa+3)f_{abc}[X_b, X_c] - [X_b, [X_b, X_a]]. \quad (6.36)$$

The equations (6.36) can also be obtained from the action (5.22) after substituting (6.24) and (6.27) into (5.22).

6.2 Invariant Gauge Fields on Homogeneous G_2-manifolds

6.2.1 Yang-Mills Equations and Explicit Calculation of the Action Functional. Considering the manifold $\mathcal{M} = \mathbb{R} \times G/H$, the Yang-Mills equations are given by

$$\ddot{X}_a = \frac{1}{6}(\kappa-1)X_a - \frac{1}{2}(\kappa+3)f_{abc}[X_b, X_c] - [X_b, [X_b, X_a]], \quad (6.37)$$

$$0 = [X_a, \dot{X}_a]. \quad (6.38)$$

These equations are the equations of motion and the Gauß constraint[2] for the action

$$\mathcal{S} = -\frac{1}{4}\int_{\mathbb{R}\times G/H} \mathrm{Tr}\left[\mathcal{F}\wedge *\mathcal{F} + \frac{\kappa}{3}e^0 \wedge \omega \wedge \mathcal{F}\wedge\mathcal{F}\right]. \quad (6.39)$$

To get the explicit form, we consider the two terms

$$\mathcal{S}_1 = \int_{\mathbb{R}\times G/H} \mathrm{Tr}(\mathcal{F}\wedge *\mathcal{F}), \qquad \mathcal{S}_2 = \int_{\mathbb{R}\times G/H} \mathrm{Tr}(\mathrm{d}\tau \wedge \omega \wedge \mathcal{F}\wedge *\mathcal{F}), \quad (6.40)$$

so that $\mathcal{S} = -\frac{1}{4}(\mathcal{S}_1 + \frac{\kappa}{3}\mathcal{S}_2)$. For $\mathcal{S}_1$ we have

$$\mathrm{Tr}(\mathcal{F}\wedge *\mathcal{F}) = \frac{1}{2}\mathrm{Tr}(\mathcal{F}_{ab}\mathcal{F}^{ab} + 2\mathcal{F}_{0a}\mathcal{F}^{0a})\mathrm{vol}_7, \quad (6.41)$$

where

$$\mathrm{vol}_7 := \mathrm{d}\tau \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \quad (6.42)$$

denotes the volume form on $\mathbb{R} \times G/H$. Now we plug in the components (6.27) for $\mathcal{F}$ to obtain

$$\begin{aligned}\mathrm{Tr}(\mathcal{F}_{ab}\mathcal{F}^{ab}) = \mathrm{Tr}\Big[& f_{ab}{}^{i} f^{abj} I_i I_j + f_{ab}{}^{i} f^{abd} I_i X_d + f^{abj} f_{ab}{}^{c} X_c I_j \\ & - f_{ab}{}^{i} I_i [X^a, X^b] - f^{abj}[X_a, X_b] I_j + f_{ab}{}^{c} f^{abd} X_c X_d \\ & - f_{ab}{}^{c} X_c [X^a, X^b] - f^{abd}[X_a, X_b] X_d + [X_a, X_b][X^a, X^b]\Big].\end{aligned} \quad (6.43)$$

[2] See also the footnote on page 38.

Since the trace is cyclic, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, we get (after pulling down all the indices)

$$\begin{aligned}\mathrm{Tr}(\mathcal{F}_{ab}\mathcal{F}_{ab}) = \mathrm{Tr}\Big[&f_{abi}f_{abj}I_iI_j + 2f_{abi}f_{abd}I_iX_d - 2f_{abi}I_i[X_a, X_b] \\ &+ f_{abc}f_{abd}X_cX_d - 2f_{abc}X_a[X_b, X_c] + [X_a, X_b][X_a, X_b]\Big].\end{aligned} \tag{6.44}$$

Now using (6.5)-(6.7) we get $f_{abi}f_{abd} = 0$ and $f_{abc}f_{abd} = \frac{1}{3}\delta_{cd}$. We further note that

$$I_i[X_a, X_b] = I_iX_aX_b - I_iX_bX_a = X_aI_iX_b - I_iX_bX_a - [X_a, I_i]X_b, \tag{6.45}$$

so that using the trace on this equation we get

$$\mathrm{Tr}(I_i[X_a, X_b]) = -\mathrm{Tr}([X_a, I_i]X_b) \tag{6.46}$$

and further

$$f_{abi}\mathrm{Tr}(I_i[X_a, X_b]) = -f_{abi}f_{aic}\mathrm{Tr}(X_cX_b) = \frac{1}{3}\mathrm{Tr}(X_bX_b). \tag{6.47}$$

Inserting (6.47) into (6.44), the equation simplifies to

$$\mathrm{Tr}(\mathcal{F}_{ab}\mathcal{F}_{ab}) = \mathrm{Tr}\Big[f_{abi}f_{abj}I_iI_j - \frac{1}{3}X_bX_b - 2f_{abc}X_a[X_b, X_c] + [X_a, X_b][X_a, X_b]\Big], \tag{6.48}$$

and with this we can write down

$$\begin{aligned}\mathcal{S}_1 = \frac{1}{2}\mathrm{Vol}(G/H)\int_{\mathbb{R}} \mathrm{d}\tau\, \mathrm{Tr}\Big[&2\dot{X}_a\dot{X}_a + f_{abi}f_{abj}I_iI_j \\ &- \frac{1}{3}X_bX_b - 2f_{abc}X_a[X_b, X_c] + [X_a, X_b][X_a, X_b]\Big].\end{aligned} \tag{6.49}$$

Next we consider the term $\mathcal{S}_2$ of the action (6.39). With $\omega = \omega_{ab}e^a \wedge e^b$ and using $\omega_{ab} = \frac{1}{2}J_{ab}$, where J is the almost complex structure, we write

$$\omega \wedge \mathcal{F} \wedge \mathcal{F} = \frac{1}{2}J_{a_1a_2}\mathcal{F}_{a_3a_4}\mathcal{F}_{a_5a_6}\varepsilon^{a_1a_2a_3a_4a_5a_6}\mathrm{vol}_6, \tag{6.50}$$

where $\mathrm{vol}_6 := e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6$ denotes the volume element on G/H. Inserting the components (6.27) of $\mathcal{F}$, we calculate

$$\begin{aligned}\mathrm{d}\tau \wedge \omega \wedge \mathcal{F} \wedge \mathcal{F} &= \frac{1}{2}J_{a_1a_2}\mathcal{F}_{a_3a_4}\mathcal{F}_{a_5a_6}\varepsilon^{a_1a_2a_3a_4a_5a_6}\mathrm{vol}_7 \\ &= \frac{1}{8}J_{a_1a_2}\Big[f_{a_3a_4i}f_{a_5a_6j}I_iI_j + f_{a_3a_4i}I_iX_c + f_{a_5a_6j}f_{a_3a_4b}X_bI_j \\ &\qquad + f_{a_3a_4b}f_{a_5a_6c}X_bX_c - f_{a_3a_4b}X_b[X_{a_5}, X_{a_6}] \\ &\qquad - f_{a_5a_6c}[X_{a_3}, X_{a_4}]X_c - f_{a_3a_4i}I_i[X_{a_5}, X_{a_6}] \\ &\qquad - f_{a_5a_6j}[X_{a_3}, X_{a_4}]I_j + [X_{a_3}, X_{a_4}][X_{a_5}, X_{a_6}]\Big]\times \\ &\qquad \times \varepsilon^{a_1a_2a_3a_4a_5a_6}\mathrm{vol}_7.\end{aligned} \tag{6.51}$$

Using the trace on this equation, the property that the trace is cyclic and that $\mathrm{Tr}(X_a I_i) = 0$, we can simplify this a bit to

$$\begin{aligned}\mathrm{Tr}(\mathrm{d}\tau \wedge \omega \wedge \mathcal{F} \wedge \mathcal{F}) = \frac{1}{8} J_{a_1 a_2} \mathrm{Tr}\Big[& f_{a_3 a_4 i} f_{a_5 a_6 j} I_i I_j + f_{a_3 a_4 b} f_{a_5 a_6 c} X_b X_c \\ & - 2 f_{a_3 a_4 b} X_b [X_{a_5}, X_{a_6}] - 2 f_{a_3 a_4 i} I_i [X_{a_5}, X_{a_6}] \\ & + [X_{a_3}, X_{a_4}][X_{a_5}, X_{a_6}] \Big] \varepsilon^{a_1 a_2 a_3 a_4 a_5 a_6} \mathrm{vol}_7.\end{aligned} \tag{6.52}$$

Now we make use of the combinatorial formula

$$J_{a_1 a_2} \varepsilon^{a_1 a_2 a_3 a_4 a_5 a_6} = 2 \left(J^{a_3 a_4} J^{a_5 a_6} - J^{a_3 a_5} J^{a_4 a_6} + J^{a_3 a_6} J^{a_4 a_5} \right). \tag{6.53}$$

By making further use of the following identities for nearly Kähler coset spaces,

$$J_{ab} f_{abc} = 0, \quad J_{ab} f_{acd} = \tilde{f}_{cdb}, \quad \tilde{f}_{acd} \tilde{f}_{bcd} = \frac{1}{3} \delta_{ab}, \tag{6.54}$$

we calculate for the different terms

$$\frac{1}{8} J_{a_1 a_2} f_{a_3 a_4 i} f_{a_5 a_6 j} \varepsilon^{a_1 a_2 a_3 a_4 a_5 a_6} = -\frac{1}{2} f_{a_5 a_6 i} f^{a_5 a_6}{}_j \tag{6.55}$$

$$\frac{1}{8} J_{a_1 a_2} f_{a_3 a_4 b} f_{a_5 a_6 c} \varepsilon^{a_1 a_2 a_3 a_4 a_5 a_6} = \frac{1}{6} \delta_{bc} \tag{6.56}$$

$$-\frac{1}{4} J_{a_1 a_2} f_{a_3 a_4 b} \varepsilon^{a_1 a_2 a_3 a_4 a_5 a_6} = 4 f^{a_5 a_6}{}_b \tag{6.57}$$

$$-\frac{1}{4} J_{a_1 a_2} f_{a_3 a_4 i} \varepsilon^{a_1 a_2 a_3 a_4 a_5 a_6} = f^{a_5 a_6}{}_i. \tag{6.58}$$

Inserting these into (6.52) and using (6.47), we can write down the second part of the action,

$$\mathcal{S}_2 = \frac{1}{2} \mathrm{Vol}(G/H) \int_{\mathbb{R}} \mathrm{d}\tau \, \mathrm{Tr}\Big[- f_{abi} f_{abj} I_i I_j + X_b X_b - 2 f_{abc} X_a [X_b, X_c] \Big], \tag{6.59}$$

from which we obtain

$$\begin{aligned}\mathcal{S} &= -\frac{1}{4} \left(\mathcal{S}_1 + \frac{\kappa}{3} \mathcal{S}_2 \right) \\ &= -\frac{1}{4} \mathrm{Vol}(G/H) \int_{\mathbb{R}} \mathrm{d}\tau \, \mathrm{Tr}\Big[\dot{X}_a \dot{X}_a + V(X) \Big]\end{aligned} \tag{6.60}$$

with a potential

$$\begin{aligned}V(X) = \frac{1}{6}(3 - \kappa) f_{abi} f_{abj} I_i I_j - \frac{1}{6}(1 - \kappa) X_a X_a \\ - \frac{1}{3}(3 + \kappa) f_{abc} X_a [X_b, X_c] + \frac{1}{2} [X_a, X_b][X_a, X_b].\end{aligned} \tag{6.61}$$

The Euler-Lagrange equations for this matrix-model action are (6.37).

6.2.2 Solution of the G-invariance Condition. The solution to the G-invariance condition (6.25) says that the X_a must transform in the six-dimensional representation $\mathcal{R}$ of H which arises in the decompositon (6.21),

$$\mathrm{Ad}\,(G)|_H = \mathrm{Ad}\,(H) \oplus \mathcal{R}, \tag{6.62}$$

of the adjoint of G restricted to H, i.e. $(\mathcal{R}(I_i))_a{}^b = f_{ia}{}^b$. It is real but reducible[3] and decomposes into complex irreducible parts as

$$\mathcal{R} = \sum_{p=1}^{q} \mathcal{R}_p \oplus \sum_{p=1}^{q} \overline{\mathcal{R}}_p, \tag{6.63}$$

with $\sum_{p=1}^{q} \dim \mathcal{R}_p = 3$. This is the same H-representation as furnished by the I_a. Hence, for each irreducible representation $\mathcal{R}_p$ one can find complex linear combinations $I^{(p)}_{\alpha_p}$ of the I_a, with $\alpha_p = 1, \dots, \dim \mathcal{R}_p$, such that

$$[I_i, I^{(p)}_{\alpha_p}] = f_{i\alpha_p}{}^{\beta_p} I^{(p)}_{\beta_p} \tag{6.64}$$

close among themselves for each p. In the absence of a condition on $[X_a, X_b]$, the X_a appear linearly and thus may always be multiplied by a common factor ϕ_p inside each irreducible representation $\mathcal{R}_p$. By Schur's lemma[4] this is the only freedom, i.e.

$$X^{(p)}_{\alpha_p} = \phi_p I^{(p)}_{\alpha_p} \quad \text{with} \quad \phi_p \in \mathbb{C}, \quad \alpha_p = 1, \dots, \dim \mathcal{R}_p, \tag{6.65}$$

is the unique solution to the G-invariance condition inside $\mathcal{R}_p$. The six antihermitian matrices X_a are then constructed via

$$\{X_a\} = \left\{ \frac{1}{2} \left(X^{(p)}_{\alpha_p} - \overline{X}^{(p)}_{\alpha_p} \right), \frac{1}{2\mathrm{i}} \left(X^{(p)}_{\alpha_p} + \overline{X}^{(p)}_{\alpha_p} \right) \right\} \tag{6.66}$$

and will depend on q complex functions $\phi_q(\tau)$. The same holds for any G-representation $\mathcal{D}$ of ad (G).

For computations, we choose a basis in $\mathfrak{g}$ such that the first $\dim(\mathcal{R}_1)$ generators I_{α_1} span $\mathcal{R}_1$, the next $\dim(\mathcal{R}_2)$ generators I_{α_2} span $\mathcal{R}_2$ etc., and the last $\dim(H)$ generators span $\mathfrak{h}$. Such a basis decomposes into the said blocks. Fusing all irreducible blocks and ad (H) together again, we obtain a realization of I_i, I_a and X_a as matrices in ad (G).

[3] Let G be a group with a representation V. A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under G. A representation V is called *irreducible*, if there is no proper nonzero invariant subspace W of V.

[4] **Lemma** (Schur, [46]). If V and W are irreducible representations of G and and $\phi : V \to W$ is a G-module homomorphism, then

(i) Either ϕ is a G-module homomorphism, or $\phi = 0$.

(ii) If $V = W$, then $\phi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$, I the identity.

Since G is the gauge group, these matrices enter in the action (6.39). However, for calculations it is more convenient to take a smaller G-representation $\mathcal{D}$. This affects only the normalization of the trace,

$$\mathrm{Tr}_{\mathcal{D}}(I_A I_B) = -\chi_{\mathcal{D}}\delta_{AB}, \tag{6.67}$$

where the *Dynkin index* $\chi_{\mathcal{D}}$ depends on the representation used. We normalize our generators such that $\chi_{\mathrm{ad}\,(G)} = 1$, and choose $\mathcal{D}$ in all cases such that $\chi_{\mathcal{D}} = \frac{1}{6}$ (see also appendix D). With this, the constant term in the action (6.60) evaluates to

$$\frac{1}{6}(3-\kappa)f_{iab}f_{jab}\mathrm{Tr}_{\mathcal{D}}(I_i I_j) = \frac{1}{36}(\kappa-3)f_{iab}f_{iab} = \frac{1}{18}(\kappa-3). \tag{6.68}$$

Chapter 7

Yang-Mills Equations and Solutions on Special G_2-Manifolds

We specialize the results of the previous chapter to the manifolds $\mathbb{R}\times SU(3)/U(1)\times U(1)$ and $\mathbb{R}\times Sp(2)/Sp(1)\times U(1)$, which both possess a G_2-structure. Further we present some solutions to the Yang-Mills equation on these manifolds. This material can also be found in [10].

7.1 Yang-Mills Fields on $\mathbb{R}\times SU(3)/U(1)\times U(1)$

7.1.1 Coset Space Generators. Expressing the X_a in terms of the Lie algebra generators I_a and complex functions ϕ_1, ϕ_2, ϕ_3, we get the following relations:

$$\begin{aligned} X_1 &= \mathrm{Re}(\phi_1)I_1 - \mathrm{Im}(\phi_1)I_2, \quad X_2 = \mathrm{Im}(\phi_1)I_1 + \mathrm{Re}(\phi_1)I_2, \\ X_3 &= \mathrm{Re}(\phi_2)I_3 + \mathrm{Im}(\phi_2)I_4, \quad X_4 = -\mathrm{Im}(\phi_2)I_3 + \mathrm{Re}(\phi_2)I_4, \\ X_5 &= \mathrm{Re}(\phi_3)I_5 - \mathrm{Im}(\phi_3)I_6, \quad X_6 = \mathrm{Im}(\phi_3)I_5 + \mathrm{Re}(\phi_3)I_6. \end{aligned} \tag{7.1}$$

Because of (6.67), we may choose the generators I_A to be normalized as

$$\mathrm{Tr}(I_A I_B) = -\frac{1}{6}\delta_{AB}. \tag{7.2}$$

Explicit form of the matrices (7.1) can be computed by using the generators given in appendix D.2. These generators also satisfy (7.2).

7.1.2 Explicit Calculation of the Lagrangian. We now substitute the equations (7.1) into the action (6.60). The kinetic term evaluates to

$$\begin{aligned} \mathrm{Tr}(\dot{X}_a\dot{X}_a) &= \mathrm{Tr}\Big[|\dot{\phi}_1|^2 I_1I_1 + |\dot{\phi}_1|^2 I_2I_2 + |\dot{\phi}_2|^2 I_3I_3 \\ &\qquad + |\dot{\phi}_2|^2 I_4I_4 + |\dot{\phi}_3|^2 I_5I_5 + |\dot{\phi}_3|^2 I_6I_6\Big] \\ &= -\frac{1}{3}\left(|\dot{\phi}_1|^2 + |\dot{\phi}_2|^2 + |\dot{\phi}_3|^2\right), \end{aligned} \tag{7.3}$$

such that

$$\frac{1}{6}(\kappa-1)\mathrm{Tr}(X_aX_a) = \frac{1}{18}(1-\kappa)\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right). \tag{7.4}$$

Next we calculate the cubic terms $f_{abc}\mathrm{Tr}(X_a[X_b, X_c])$. Since most of the structure constants are zero, and because of the permutation symmetries of the equation, we only need to compute the combinations (abc) equal to (154), (136), (246) and (235). Since for every term we calculate we may permute the indices in 3! ways, we get an additional factor 3!. Then we have

$$\begin{aligned}3!f_{154}\mathrm{Tr}(X_1[X_5, X_4]) &= \sqrt{3}\mathrm{Tr}\Big[X_1[\mathrm{Re}(\phi_3)I_5 - \mathrm{Im}(\phi_3)I_6, -\mathrm{Im}(\phi_2)I_3 + \mathrm{Re}(\phi_2)I_4]\Big] \\ &= \frac{1}{2}\mathrm{Tr}\Big[(\mathrm{Re}(\phi_1)I_1 - \mathrm{Im}(\phi_1)I_2)\times \\ &\qquad \times \big\{(\mathrm{Re}(\phi_3)\mathrm{Im}(\phi_2) + \mathrm{Im}(\phi_3)\mathrm{Re}(\phi_2))I_2 \\ &\qquad\quad + (\mathrm{Re}(\phi_3)\mathrm{Re}(\phi_2) - \mathrm{Im}(\phi_3)\mathrm{Im}(\phi_2))I_1\big\}\Big] \\ &= \frac{1}{2}\mathrm{Tr}\Big[\,(\mathrm{Re}(\phi_1)\mathrm{Re}(\phi_2)\mathrm{Re}(\phi_3) - \mathrm{Re}(\phi_1)\mathrm{Im}(\phi_2)\mathrm{Im}(\phi_3))\,I_1I_1 \\ &\qquad - (\mathrm{Im}(\phi_1)\mathrm{Im}(\phi_2)\mathrm{Re}(\phi_3) + \mathrm{Im}(\phi_1)\mathrm{Re}(\phi_2)\mathrm{Im}(\phi_3))\,I_2I_2\Big],\end{aligned} \tag{7.5}$$

where we made use of the Lie algebra identities and used $\mathrm{Tr}(I_AI_B) = 0$ for $A \neq B$ in the last step. In the same manner we evaluate the other cubic terms,

$$\begin{aligned}3!f_{136}\mathrm{Tr}(X_1[X_3, X_6]) &= \frac{1}{2}\mathrm{Tr}\Big[(\mathrm{Re}(\phi_1)\mathrm{Re}(\phi_2)\mathrm{Re}(\phi_3) - \mathrm{Re}(\phi_1)\mathrm{Im}(\phi_2)\mathrm{Im}(\phi_3))I_1I_1 \\ &\qquad - (\mathrm{Im}(\phi_1)\mathrm{Re}(\phi_2)\mathrm{Im}(\phi_3) + \mathrm{Im}(\phi_1)\mathrm{Im}(\phi_2)\mathrm{Re}(\phi_3))I_2I_2\Big]\end{aligned} \tag{7.6}$$

$$\begin{aligned}3!f_{246}\mathrm{Tr}(X_2[X_4, X_6]) &= -\frac{1}{2}\mathrm{Tr}\Big[(\mathrm{Im}(\phi_1)\mathrm{Im}(\phi_2)\mathrm{Re}(\phi_3) + \mathrm{Im}(\phi_1)\mathrm{Re}(\phi_2)\mathrm{Im}(\phi_3))I_1I_1 \\ &\qquad + (\mathrm{Re}(\phi_1)\mathrm{Im}(\phi_2)\mathrm{Im}(\phi_3) - \mathrm{Re}(\phi_1)\mathrm{Re}(\phi_2)\mathrm{Re}(\phi_3))I_2I_2\Big]\end{aligned} \tag{7.7}$$

$$\begin{aligned}3!f_{235}\mathrm{Tr}(X_2[X_3, X_5]) &= \frac{1}{2}\mathrm{Tr}\Big[(-\mathrm{Im}(\phi_1)\mathrm{Re}(\phi_2)\mathrm{Im}(\phi_3) - \mathrm{Im}(\phi_1)\mathrm{Im}(\phi_2)\mathrm{Re}(\phi_3))I_1I_1 \\ &\qquad + (\mathrm{Re}(\phi_1)\mathrm{Re}(\phi_2)\mathrm{Re}(\phi_3) - \mathrm{Re}(\phi_1)\mathrm{Im}(\phi_2)\mathrm{Im}(\phi_3))I_2I_2\Big].\end{aligned} \tag{7.8}$$

Adding up (7.5)-(7.8), using the trace normalization (7.2) and collecting terms, we get for the cubic term in the action (6.60)

$$\begin{aligned}-\frac{1}{3}(\kappa+3)f_{abc}\mathrm{Tr}(X_a[X_b, X_c]) &= \frac{1}{9}(\kappa+3)\mathrm{Re}(\phi_1\phi_2\phi_3) \\ &= \frac{1}{18}(\kappa+3)(\phi_1\phi_2\phi_3 + \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3).\end{aligned} \tag{7.9}$$

For the quartic terms $[X_a, X_b][X_a, X_b]$ we have to calculate 15 terms, which are given by $[X_a, X_b][X_a, X_b]$ for $1 \le a < b \le 6$. Considering the symmetries in the definition of the X_a's, (7.1), this number can be further reduced, so that we only have to evaluate 3 double commutators, which are for example $[X_1, X_2]^2$, $[X_1, X_3]^2$ and $[X_1, X_5]^2$:

$$\begin{aligned}
\mathrm{Tr}([X_1, X_2]^2) &= \mathrm{Tr}\Big[[\mathrm{Re}(\phi_1)I_1 - \mathrm{Im}(\phi_1)I_2, \mathrm{Im}(\phi_1)I_1 + \mathrm{Re}(\phi_1)I_2]^2\Big] \\
&= \mathrm{Tr}\left[\frac{1}{3}(\mathrm{Re}(\phi_1)^2 + \mathrm{Im}(\phi_1)^2)I_7 I_7\right] \\
&= -\frac{1}{18}|\phi_1|^4, && (7.10) \\
\mathrm{Tr}([X_1, X_3]^2) &= \mathrm{Tr}\Big[[\mathrm{Re}(\phi_1)I_1 - \mathrm{Im}(\phi_1)I_2, \mathrm{Re}(\phi_2)I_3 + \mathrm{Im}(\phi_2)I_4]^2\Big] \\
&= \frac{1}{6}\mathrm{Tr}\Big[(\mathrm{Re}(\phi_1)\mathrm{Re}(\phi_2) - \mathrm{Im}(\phi_1)\mathrm{Im}(\phi_2))I_6 \\
&\qquad - (\mathrm{Re}(\phi_1)\mathrm{Im}(\phi_2) + \mathrm{Im}(\phi_1)\mathrm{Re}(\phi_2))I_5\Big]^2 \\
&= \frac{1}{6}\mathrm{Tr}\Big[\mathrm{Re}^2(\phi_1\phi_2)I_6 I_6 + \mathrm{Im}^2(\phi_1\phi_2)I_5 I_5\Big] \\
&= -\frac{1}{36}|\phi_1\phi_2|^2, && (7.11) \\
\mathrm{Tr}([X_1, X_5]^2) &= \mathrm{Tr}\Big[[\mathrm{Re}(\phi_1)I_1 - \mathrm{Im}(\phi_1)I_2, \mathrm{Re}(\phi_3)I_5 - \mathrm{Im}(\phi_3)I_6]^2\Big] \\
&= \frac{1}{6}\mathrm{Tr}\Big[\mathrm{Re}^2(\phi_1\phi_3)I_4 I_4 + \mathrm{Im}^2(\phi_1\phi_3)I_3 I_3\Big] \\
&= -\frac{1}{36}|\phi_1\phi_3|^2. && (7.12)
\end{aligned}$$

Now because of the symmetries mentioned above, we also know the traces of the other commutators:

$$\mathrm{Tr}([X_1, X_6]^2) = \mathrm{Tr}([X_1, X_5]^2) = -\frac{1}{36}|\phi_1\phi_3|^2, \tag{7.13}$$
$$\mathrm{Tr}([X_2, X_3]^2) = \mathrm{Tr}([X_2, X_4]^2) = -\frac{1}{36}|\phi_1\phi_2|^2, \tag{7.14}$$
$$\mathrm{Tr}([X_2, X_5]^2) = \mathrm{Tr}([X_2, X_6]^2) = -\frac{1}{36}|\phi_1\phi_3|^2, \tag{7.15}$$
$$\mathrm{Tr}([X_3, X_4]^2) = -\frac{1}{18}|\phi_2|^4, \tag{7.16}$$
$$\mathrm{Tr}([X_3, X_5]^2) = \mathrm{Tr}([X_3, X_6]^2) = -\frac{1}{36}|\phi_2\phi_3|^2, \tag{7.17}$$
$$\mathrm{Tr}([X_4, X_5]^2) = \mathrm{Tr}([X_4, X_6]^2) = -\frac{1}{36}|\phi_2\phi_3|^2, \tag{7.18}$$
$$\mathrm{Tr}([X_5, X_6]^2) = -\frac{1}{18}|\phi_3|^4. \tag{7.19}$$

Adding up the equations (6.68), (7.4), (7.9) and (7.10)-(7.19), we obtain the Lagrangian

$$
\begin{aligned}
18\mathcal{L} = (3-\kappa) &+ 6\big(|\dot{\phi}_1|^2 + |\dot{\phi}_2|^2 + |\dot{\phi}_3|^2\big) \\
&+ (\kappa-1)\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right) \\
&- (\kappa+3)\big(\phi_1\phi_2\phi_3 + \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3\big) \\
&+ |\phi_1\phi_2|^2 + |\phi_1\phi_3|^2 + |\phi_2\phi_3|^2 \\
&+ |\phi_1|^4 + |\phi_2|^4 + |\phi_3|^4.
\end{aligned} \tag{7.20}
$$

7.1.3 Equations of Motion. The equations of motions for the gauge fields on $\mathbb{R} \times SU(3)/U(1) \times U(1)$ can be obtained by plugging (7.1) into (6.37) and (6.38). Since the calculation is very similar to the calculation for the Lagrangian, we only state the results. We consider the equations of motion for X_1, which yield the same result as the equations for X_2. The equations for X_3, X_4, X_5, X_6 then are obtained by cyclic permutation of the ϕ_i's. Now for the commutator we calculate

$$
\begin{aligned}
f_{1bc}[X_b, X_c] &= \frac{1}{\sqrt{3}}([X_3, X_6] + [X_5, X_4]) \\
&= \frac{1}{3}(\mathrm{Re}(\phi_2\phi_3)I_1 + \mathrm{Im}(\phi_2\phi_3)I_2).
\end{aligned} \tag{7.21}
$$

For the double commutators we get

$$
[X_2,[X_2,X_1]] = -\frac{1}{3}|\phi_1|^2 X_1, \qquad [X_3,[X_3,X_1]] = -\frac{1}{12}|\phi_2|^2 X_1, \tag{7.22}
$$

$$
[X_4,[X_4,X_1]] = -\frac{1}{12}|\phi_2|^2 X_1, \qquad [X_5,[X_5,X_1]] = -\frac{1}{12}|\phi_3|^2 X_1, \tag{7.23}
$$

$$
[X_6,[X_6,X_1]] = -\frac{1}{12}|\phi_3|^2 X_1. \tag{7.24}
$$

Plugging this back into the general equations of motion (6.37), we get

$$
6\ddot{X}_1 = (\kappa-1)X_1 - (\kappa+3)(\mathrm{Re}(\phi_2\phi_3)I_1 + \mathrm{Im}(\phi_2\phi_3)I_2) + (2|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2)X_1. \tag{7.25}
$$

Now using $X_1 = \mathrm{Re}(\phi_1)I_1 - \mathrm{Im}(\phi_1)I_2$ from (7.1) and comparing the coefficients for I_1 and I_2, we arrive at the following equations of motion (the second and third equation are obtained by cyclic permutations):

$$
\begin{aligned}
6\ddot{\phi}_1 &= (\kappa-1)\phi_1 - (\kappa+3)\bar{\phi}_2\bar{\phi}_3 + (2|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2)\phi_1, \\
6\ddot{\phi}_2 &= (\kappa-1)\phi_2 - (\kappa+3)\bar{\phi}_3\bar{\phi}_1 + (|\phi_1|^2 + 2|\phi_2|^2 + |\phi_3|^2)\phi_2, \\
6\ddot{\phi}_3 &= (\kappa-1)\phi_3 - (\kappa+3)\bar{\phi}_1\bar{\phi}_2 + (|\phi_1|^2 + |\phi_2|^2 + 2|\phi_3|^2)\phi_3.
\end{aligned} \tag{7.26}
$$

In the same manner we calculate the constraints (6.38). These are given by

$$
\phi_1\dot{\bar{\phi}}_1 - \dot{\phi}_1\bar{\phi}_1 = \phi_2\dot{\bar{\phi}}_2 - \dot{\phi}_2\bar{\phi}_2 = \phi_3\dot{\bar{\phi}}_3 - \dot{\phi}_3\bar{\phi}_3. \tag{7.27}
$$

The equations (7.26) are the Euler-Lagrange equations for the Lagrangian (7.20) obtained from (6.39) after fixing the gauge $\mathcal{A}_0 = 0$.

7.1.4 Zero-Energy Critical Points of the Potential. We may write the equations of motion (7.26) as

$$6\ddot{\phi}_i = \frac{\partial V}{\partial \bar{\phi}_i}, \tag{7.28}$$

so that they may be interpreted to describe the motion of a single particle in $\mathbb{C}^3$ under the influence of the inverted quartic potential $-V$ with

$$\begin{aligned} V = -(\kappa - 3) + (\kappa - 1)(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2) \\ + (|\phi_1|^4 + |\phi_2|^4 + |\phi_3|^4) \\ - (\kappa + 3)(\phi_1\phi_2\phi_3 + \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3) \\ + |\phi_1\phi_2|^2 + |\phi_2\phi_3|^2 + |\phi_3\phi_1|^2. \end{aligned} \tag{7.29}$$

An alternative interpretation of the equations of motion is that they describe the dynamics of three (identical) particles in the complex plane, with an external potential given by the negative of the first two lines in (7.29) and two- and three-body interactions in the third and fourth line.

The potential (7.29) is invariant under permutations of the ϕ_i as well as under $U(1) \times U(1)$ transformations

$$(\phi_1, \phi_2, \phi_3) \to (\mathrm{e}^{\mathrm{i}\alpha_1}\phi_1, \mathrm{e}^{\mathrm{i}\alpha_2}\phi_2, \mathrm{e}^{-\mathrm{i}(\alpha_1+\alpha_2)}\phi_3), \tag{7.30}$$

which include the 3-symmetry $\phi_i \mapsto \mathrm{e}^{2\pi\mathrm{i}/3}\phi_i$. Such a transformation may be used to align the phases of the ϕ_i, i.e $\arg(\phi_1) = \arg(\phi_2) = \arg(\phi_3)$. These phases only enter in the cubic term of the potential, which is proportional to $\cos\left(\sum_i \arg\phi_i\right)$. Therefore, the extrema of V are attained at $\sum_i \arg\phi_i = 0$ or π, and so, employing (7.30), we may take $\phi_i \in \mathbb{R}$ in our search for them. Furthermore, the Noether charges of the $U(1) \times U(1)$ symmetry (7.30) are the differences $l_i - l_j$ of the "angular momenta"

$$l_i := \dot{\phi}_i\bar{\phi}_i - \phi_i\dot{\bar{\phi}}_i. \tag{7.31}$$

Hence, the constraints (7.27) may be interpreted as putting these charges to zero. Note, however, that the individual angular momenta are not conserved, since

$$\begin{aligned} \dot{l}_i &= \bar{\phi}_i\ddot{\phi}_i - \ddot{\bar{\phi}}_i\phi_i \\ &= \frac{1}{6}(\kappa + 3)(\phi_1\phi_2\phi_3 - \bar{\phi}_1\bar{\phi}_2\bar{\phi}_3). \end{aligned} \tag{7.32}$$

Finite-action solutions $\phi_i(\tau)$ must interpolate between critical points with zero potential,

$$\lim_{\tau\to\pm\infty} \phi_i(\tau) =: \phi_i^{\pm} \quad \text{and} \quad (\phi_1^{\pm}, \phi_2^{\pm}, \phi_3^{\pm}) \in \{\hat{\phi}\} \quad \text{with} \quad V(\hat{\phi}) = 0 = \mathrm{d}V(\hat{\phi}). \tag{7.33}$$

We now construct a complete list $\{\hat{\phi}_i^\alpha\}_{i=1,2,3}$, of the critical points, where $\alpha = A, A', B, C$ labels the different types. First we assume (without loss of generality, since we can employ the permutation symmetry) that $\phi_1 = 0$. From the equations of motion (7.26) it now follows that without loss of generality $\phi_2 = 0$. The remaining equation has the solutions

$$(\hat{\phi}_1^B, \hat{\phi}_2^B, \hat{\phi}_3^B) = (0,0,0), \qquad (\phi_1, \phi_2, \phi_3) = (0, 0, \sqrt{(1-\kappa)/2}). \tag{7.34}$$

The first solution already has $V(\hat{\phi}^1) = 0$. For the second solution this condition implies $\kappa = -1 \pm 2\sqrt{3}$. Since we may choose ϕ_3 to be real, we get $\kappa = -1 - 2\sqrt{3}$ and hence

$$(\hat{\phi}_1^C, \hat{\phi}_2^C, \hat{\phi}_3^C) = \left(0, 0, \sqrt{1+\sqrt{3}}\right). \tag{7.35}$$

We have seen that a single vanishing ϕ_i implies a second ϕ_j, $i \neq j$ to be zero. So now we may consider the case of not exactly two ϕ_i being zero, and then the equation

$$\begin{aligned}
0 &= \frac{\partial V}{\partial \bar{\phi}_1} + \frac{\partial V}{\partial \bar{\phi}_2} + \frac{\partial V}{\partial \bar{\phi}_3} \\
&= (\kappa - 1)(\phi_1 + \phi_2 + \phi_3) - (\kappa + 3)(\bar{\phi}_1\bar{\phi}_2 + \bar{\phi}_1\bar{\phi}_3 + \bar{\phi}_2\bar{\phi}_3) \\
&\quad + (|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2)(\phi_1 + \phi_2 + \phi_3) \\
&\quad + |\phi_1|^2\phi_1 + |\phi_2|^2\phi_2 + |\phi_3|^3\phi_3.
\end{aligned} \tag{7.36}$$

If we had exactly two vanishing ϕ_i's, this equation would reduce to the case discussed above. For now, consider the case $\kappa \neq -3$. Then the coefficient of the $(\kappa+3)$-term and the coefficient of the $(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2)$-term have to vanish independently, which generates the two conditions

$$\phi_1 + \phi_2 + \phi_3 = 0 \quad \text{and} \quad \phi_1\phi_2 + \phi_2\phi_3 + \phi_1\phi_3 = 0, \tag{7.37}$$

with the solution

$$\phi_1 = \exp(2\pi\mathrm{i}/3)\phi_3, \quad \phi_2 = \exp(4\pi\mathrm{i}/3)\phi_3. \tag{7.38}$$

Using these values, the equation (7.36) is satisfied and we can insert ϕ_1 and ϕ_2 into $\frac{\partial V}{\partial \bar{\phi}_3} = 0$ to get the equation

$$(\kappa - 1)\phi_3 - (\kappa + 3)\bar{\phi}_3^2 + 4|\phi_3|^2\phi_3 = 0, \tag{7.39}$$

which was treated in [9]. Using the results from there, we get the points

$$(\hat{\phi}_1^B, \hat{\phi}_2^B, \hat{\phi}_3^B) = (0,0,0), \tag{7.40}$$
$$(\hat{\phi}_1^A, \hat{\phi}_2^A, \hat{\phi}_3^A) = (\mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{4\pi\mathrm{i}/3}, 1), \tag{7.41}$$
$$(\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3) = \rho(\mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{4\pi\mathrm{i}/3}, 1) \qquad \text{with} \quad \rho = \frac{1}{4}(\kappa - 1). \tag{7.42}$$

type	ϕ_1	ϕ_2	ϕ_3	κ
A	1	1	1	any
A'	$\mathrm{e}^{\mathrm{i}\alpha}$	$\mathrm{e}^{\mathrm{i}\alpha}$	$\mathrm{e}^{\mathrm{i}\alpha}$	-3
B	0	0	0	$+3$
C	0	0	$\sqrt{1+\sqrt{3}}$	$-1-2\sqrt{3}$

Table 7.1: List of critical points

type	eigenvalues of V''					
A	0	0	$3(\kappa+3)$	$2(\kappa+4)$	$2(\kappa+4)$	$5-\kappa$
A'	0	0	0	2	2	8
B	2	2	2	2	2	2
C	0	γ_-	γ_-	γ_+	γ_+	$4(1+\sqrt{3})$

Table 7.2: Eigenvalues of V'' with $\gamma_\pm = -(1+\sqrt{3}) \pm 2\sqrt{2(\sqrt{3}-1)}$.

The last point has $V = 0$ only if $\kappa = -3$ (which is a contradiction to the assumption above) or $\kappa = 5$. But choosing $\kappa = 5$ we get $\rho = 1$ and we are back at the critical point $(\hat{\phi}_1^A, \hat{\phi}_2^A, \hat{\phi}_3^A) = (\mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{4\pi\mathrm{i}/3}, 1)$. Note that by employing the $U(1) \times U(1)$-symmetry (7.30), this point is equivalent to the point $(1, 1, 1)$.

Last, we have to consider the special case $\kappa = -3$. Then the potential V is given by

$$
\begin{aligned}
V = 6 &- 4(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2) \\
&+ |\phi_1\phi_2|^2 + |\phi_2\phi_3|^2 + |\phi_1\phi_3|^2 + |\phi_1|^4 + |\phi_2|^4 + |\phi_3|^4,
\end{aligned}
\tag{7.43}
$$

which is rotationally symmetric. Considering the equations for the critical points,

$$
0 = \frac{\partial V}{\partial \bar{\phi}_1} = -4\phi_1 + \phi_1\big(2|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\big) \tag{7.44}
$$

and cyclic permutations of the ϕ_i, we get for $\phi_i \neq 0$ the condition

$$
2|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = 4 \tag{7.45}
$$

and its cyclic permutations. The solution to these equations is $|\phi_i| = 1$. Since in the special case of $\kappa = -3$ all phase dependence disappears and the $U(1) \times U(1)$-symmetry (7.30) is extended to $U(1)^3$, we get the critical point

$$
(\hat{\phi}_1^{A'}, \hat{\phi}_2^{A'}, \hat{\phi}_3^{A'}) = (\mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\alpha}), \quad \alpha \in \mathbb{R}, \qquad \text{for} \quad \kappa = -3. \tag{7.46}
$$

Since any possibilities for values of the ϕ_i's can be reduced to the cases we discussed, we get a complete list of the critical points (see the tables 7.1 and 7.2).

The zero modes of V'' are enforced by the symmetries; their number indicates the dimension of the critical manifold in $\mathbb{C}^3$. A critical point is marginally stable only when V'' has no positive eigenvalues. At the critical points $\dot{l}_i = 0$ is guaranteed, hence the product $\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3$ has to be real unless $\kappa = -3$. As noted above, the latter value is special because all phase dependence disappears, and the symmetry (7.30) is enhanced to $U(1)^3$. We will not consider this special situation (type A') further.

7.1.5 Some Solutions. Finite-action trajectories $\phi_i(\tau)$ require the conserved Newtonian energy to vanish,

$$E := 6\left(|\dot{\phi}_1|^2 + |\dot{\phi}_2|^2 + |\dot{\phi}_3|^2\right) - V(\phi_1, \phi_2, \phi_3) \stackrel{!}{=} 0. \tag{7.47}$$

They can be of two types: Either $\phi_i^+ \neq \phi_i^-$ (kink), or $\phi_i^+ = \phi_i^-$ (bounce). Since this choice occurs for each value of $i = 1, 2, 3$, mixed solutions are possible. We now present some special cases.

Transverse kinks at $-3 < \kappa < 3$. The two-dimensional type A critical manifold exists for any value of κ, so one may try to find trajectories connecting two critical points of type A. As a particularly symmetric choice we wish to interpolate

$$(\phi_i^-) = (1, \mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{-2\pi\mathrm{i}/3}) \quad \rightarrow \quad (\phi_i^+) = (\mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{-2\pi\mathrm{i}/3}, 1). \tag{7.48}$$

The three independent conserved quantities $(E, l_i - l_j)$ do not suffice to integrate the equations of motion (7.26), so generically one has to resort to numerical methods. Zero-enery 'transverse' kinks can be found in the range $\kappa \in (-3, 3)$, as shown in figure 7.1 for $\kappa = -3, -2, -1, 0, 1, 2$ and in figure 7.2 for $\kappa = 2.5, 3$. Since $\phi_2(\tau) = \mathrm{e}^{2\pi\mathrm{i}/3}\phi_1(\tau) = \mathrm{e}^{-2\pi\mathrm{i}/3}\phi_3(\tau)$, the constraints (7.27) are resolved. The action of the symmetry transformations (7.30) on the solutions generates a two-parameter family of such 'transverse' kinks.

At the magical value of $\kappa = -1$ the trajectories become straight, and the solution analytic:

$$\begin{aligned}
\phi_1(\tau) &= \left(\frac{1}{4} + \mathrm{i}\frac{\sqrt{3}}{4}\right) + \left(-\frac{3}{4} + \mathrm{i}\frac{\sqrt{3}}{4}\right)\tanh\left(\frac{\tau - \tau_0}{2}\right), \\
\phi_2(\tau) &= -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\tanh\left(\frac{\tau - \tau_0}{2}\right), \\
\phi_3(\tau) &= \left(\frac{1}{4} - \mathrm{i}\frac{\sqrt{3}}{4}\right) + \left(\frac{3}{4} + \mathrm{i}\frac{\sqrt{3}}{4}\right)\tanh\left(\frac{\tau - \tau_0}{2}\right).
\end{aligned} \tag{7.49}$$

Radial kinks at $\kappa = 3$. For this value of κ the critical point at the origin is degenerate with $(1, 1, 1)$ and its symmetry orbits. Therefore, we can connect any type A critical point to the unique type B point via 'radial' kinks, such as

$$
\begin{aligned}
\phi_1(\tau) &= \frac{1}{2}\left(1 + \tanh\left(\frac{\tau - \tau_0}{2\sqrt{3}}\right)\right), \\
\phi_2(\tau) &= \left(-\frac{1}{4} + \mathrm{i}\frac{\sqrt{3}}{4}\right)\left(1 + \tanh\left(\frac{\tau - \tau_0}{2\sqrt{3}}\right)\right), \\
\phi_3(\tau) &= \left(-\frac{1}{4} - \mathrm{i}\frac{\sqrt{3}}{4}\right)\left(1 + \tanh\left(\frac{\tau - \tau_0}{2\sqrt{3}}\right)\right),
\end{aligned}
\tag{7.50}
$$

which connects

$$
(0,0,0) \quad \rightarrow \quad (1, \mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{-2\pi\mathrm{i}/3}) \tag{7.51}
$$

in a 3-symmetric fashion and is also marked in the right plot of figure 7.2. It is the limiting case of the transverse kinks for $\kappa \rightarrow 3$. In the other limit, $\kappa \rightarrow -3$, the particles move infinitely slow on the degenerate unit circle $|\phi| = 1$.

Bounces at $\kappa < -3$ and $3 < \kappa < 5$. In the range $\kappa \in (-\infty, -3) \cup (3, 5)$ finite-action bounce solutions must exist, in the form

$$
\phi_k(\tau) = \mathrm{e}^{2\pi\mathrm{i}(k-1)/3} f_\kappa(\tau) \quad \text{with} \quad f_\kappa(\pm\infty) = 1 \quad \text{and} \quad f_\kappa(0) = \frac{1}{6}\left(\kappa - 3 + \sqrt{\kappa^2 - 9}\right), \tag{7.52}
$$

where $f_\kappa(\tau)$ is a real function, so the trajectories are straight. Figure 7.3 shows the trajectories for $\kappa = -4$ and $\kappa = 4$.

Radial bounce/kink at $\kappa = -1 - 2\sqrt{3}$. If we put $\phi_1(\tau) = \phi_2(\tau) \equiv 0$ at this κ value, the remaining function is governed by the rotationally symmetric potential

$$
V(0,0,\phi_3) = 2\big(2 + \sqrt{3}\big) - \big(1 + \sqrt{3}\big)|\phi_3|^2 + |\phi_3|^4, \tag{7.53}
$$

so that for the equations of motion we get a kink-type solution

$$
\phi_3(\tau) = \mathrm{e}^{\mathrm{i}\alpha}\sqrt{1 + \sqrt{3}}\tanh\left(\sqrt{\frac{1 + \sqrt{3}}{6}}\,\tau\right). \tag{7.54}
$$

This interpolates between antipodal type C critical points via point B,

$$
\left(0, 0, -\mathrm{e}^{\mathrm{i}\alpha}\sqrt{1 + \sqrt{3}}\right) \quad \rightarrow \quad \left(0, 0, \mathrm{e}^{\mathrm{i}\alpha}\sqrt{1 + \sqrt{3}}\right). \tag{7.55}
$$

7.2 Yang-Mills Fields on $\mathbb{R} \times Sp(2)/Sp(1) \times U(1)$

7.2.1 Coset Space Generators. We express X_a in terms of the I_a and two complex functions ϕ_1 and ϕ_2,

$$\begin{aligned} X_1 &= -\mathrm{Im}(\phi_1) I_1 + \mathrm{Re}(\phi_1) I_2, \quad & X_2 &= \mathrm{Re}(\phi_1) I_1 + \mathrm{Im}(\phi_1) I_2, \\ X_3 &= \mathrm{Re}(\phi_1) I_3 + \mathrm{Im}(\phi_1) I_4, \quad & X_4 &= \mathrm{Im}(\phi_1) I_3 - \mathrm{Re}(\phi_1) I_4, \\ X_5 &= -\mathrm{Re}(\phi_2) I_5 + \mathrm{Im}(\phi_2) I_6, \quad & X_6 &= -\mathrm{Im}(\phi_2) I_5 - \mathrm{Re}(\phi_2) I_6. \end{aligned} \tag{7.56}$$

7.2.2 Equations of Motion. Inserting the generators (7.56) into the action functional (6.39), after a similar calculation as for the $\mathbb{R} \times SU(3)/U(1) \times U(1)$-case we obtain the Lagrangian

$$\begin{aligned} 18\mathcal{L} = (3-\kappa) + 12|\dot{\phi}_1|^2 + 6|\dot{\phi}_2|^2 + (\kappa-1)\big(2|\phi_1|^2 + |\phi_2|^2\big) \\ - (\kappa+3)\big(\phi_1^2\phi_2 + \bar{\phi}_1^2\bar{\phi}_2\big) + 3|\phi_1|^4 + 2|\phi_1\phi_2|^2 + |\phi_2|^4. \end{aligned} \tag{7.57}$$

The equations of motion are now given by

$$\begin{aligned} 6\ddot{\phi}_1 &= (\kappa-1)\phi_1 - (\kappa+3)\bar{\phi}_1\bar{\phi}_2 + (3|\phi_1|^2 + |\phi_2|^2)\phi_1, \\ 6\ddot{\phi}_2 &= (\kappa-1)\phi_2 - (\kappa+3)\bar{\phi}_1^2 + 2(|\phi_1|^2 + |\phi_2|^2)\phi_2, \end{aligned} \tag{7.58}$$

and the constraints (6.38) read

$$\phi_1\dot{\bar{\phi}}_1 - \dot{\phi}_1\bar{\phi}_1 = \phi_2\dot{\bar{\phi}}_2 - \dot{\phi}_2\bar{\phi}_2. \tag{7.59}$$

By identifying $\phi_1 = \phi_2 =: \phi$, we again return to the $G_2/SU(3)$-case. Note that if we identify ϕ_1 and ϕ_2 in (7.26), we also get (7.58) and (7.59). In the same manner, (7.57) is obtained from (7.20). The equations (7.58) are the Euler-Lagrange equations for the Lagrangian (7.57),

$$12\,\ddot{\phi}_1 = \frac{\partial V}{\partial \bar{\phi}_1}, \qquad 6\,\ddot{\phi}_2 = \frac{\partial V}{\partial \bar{\phi}_2}, \tag{7.60}$$

and the constraint (7.59) derives from the $U(1)$ symmetry

$$(\phi_1, \phi_2) \quad \to \quad (\mathrm{e}^{\mathrm{i}\delta}\phi_1, \mathrm{e}^{-2\mathrm{i}\delta}\phi_2) \tag{7.61}$$

of the potential

$$\begin{aligned} V = (3-\kappa) + (\kappa-1)\big(2|\phi_1|^2 + |\phi_2|^2\big) - (\kappa+3)\big(\phi_1^2\phi_2 + \bar{\phi}_1^2\bar{\phi}_2\big) \\ + 3|\phi_1|^4 + 2|\phi_1\phi_2|^2 + |\phi_2|^4. \end{aligned} \tag{7.62}$$

7.2.3 Some Solutions. The solutions to (7.58) and (7.59) form a subset of the solutions to (7.26) and (7.27), namely those where two functions coincide. Since in all examples of the previous section this can be arranged by applying a $U(1) \times U(1)$-transformation, one gets $\varphi(\tau) = \chi(\tau)$ equal to any of the functions appearing on the right-hand sides of (7.49) and (7.50) or depicted in the figures 7.1, 7.2 and 7.3, after choosing the corresponding κ-value. In addition, (7.54) translates to a solution with $\varphi \equiv 0$ and a kink χ.

7.2.4 Specialization so $S^6 = G_2/SU(3)$ and Flow Equations. By further identification

$$\phi_1 = \phi_2 = \phi_3 =: \phi \tag{7.63}$$

we resolve the constraint equations (7.27) and reduce (7.26) to the requation

$$6\,\ddot{\phi} = (\kappa-1)\phi - (\kappa+3)\bar{\phi}^2 + 4|\phi|^2\phi = \frac{1}{3}\frac{\partial V}{\partial\bar{\phi}} \tag{7.64}$$

with

$$V = (3-\kappa) + 3(\kappa-1)|\phi|^2 - (\kappa+3)\big(\phi^3+\bar{\phi}^3\big) + 6|\phi|^4. \tag{7.65}$$

The $U(1)$ symmetry (7.61) is broken to the discrete 3-symmetry and the Lagrangian (7.57) becomes

$$18\mathcal{L} = 18|\dot{\phi}|^2 + V(\phi), \tag{7.66}$$

which describes G_2-invariant gauge fields on $\mathbb{R} \times G_2/SU(3)$ (see [9]). Note that any function on the right-hand side in (7.49) and (7.50) or shown in the figures 7.1, 7.2 and 7.3 is a zero-energy solution [9]. Vice versa, any solution of (7.64) gives a special solution to the equations (7.58), (7.59) and (7.26), (7.27).

We now try to construct some analytical solutions $\phi(\tau)$ to (7.64). For simplification, we assume $\phi(\tau)$ to be a straight trajectory, which (because of the 3-symmetry transformations) can be brought into a form where either $\mathrm{Re}\phi(\tau) = \mathrm{const}$ or $\mathrm{Im}\phi(\tau) = \mathrm{const}$. Then, the vanishing of the left-hand side of Re(7.64) yields two conditions on $\mathrm{Re}\phi$ and κ, whose solutions follow a Hamiltonian flow [9]:

$$\begin{aligned}
(\kappa,\mathrm{Re}\phi) &= (-1,-1/2) &&\Rightarrow\ \sqrt{3}\mathrm{Im}\dot{\phi} = \frac{3}{4} - (\mathrm{Im}\phi)^2 &&\Leftrightarrow\ \sqrt{3}\dot{\phi} = \mathrm{i}\big(\bar{\phi}^2 - \phi\big),\\
(\kappa,\mathrm{Re}\phi) &= (-3,0) &&\Rightarrow\ \sqrt{3}\mathrm{Im}\dot{\phi} = 1 - (\mathrm{Im}\phi)^2 &&\Leftrightarrow\ \sqrt{3}\dot{\phi} = \frac{\phi}{|\phi|}\big(1-|\phi|^2\big),\\
(\kappa,\mathrm{Re}\phi) &= (-7,1) &&\Rightarrow\ \sqrt{3}\mathrm{Im}\dot{\phi} = 3 - (\mathrm{Im}\phi)^2 &&\Leftrightarrow\ \sqrt{3}\dot{\phi} = \mathrm{i}\big(\bar{\phi}^2 + 2\phi\big).
\end{aligned} \tag{7.67}$$

On the other hand, for $\mathrm{Im}\ddot{\phi} = 0$ one finds for any value of κ

$$\mathrm{Im}\phi = 0 \quad\Rightarrow\quad 6\,\mathrm{Re}\ddot{\phi} = (\kappa-1)\mathrm{Re}\phi - (\kappa+3)(\mathrm{Re}\phi)^2 + 4(\mathrm{Re}\phi)^3 = \frac{1}{3}\frac{\partial V_{\mathbb{R}}}{\partial \mathrm{Re}\phi}, \tag{7.68}$$

with

$$V_{\mathbb{R}} = (\mathrm{Re}\phi - 1)^2\left(6(\mathrm{Re}\phi)^2 - (\kappa-3)(2\mathrm{Re}\phi+1)\right). \tag{7.69}$$

This includes the gradient flow situations

$$\begin{aligned}
(\kappa,\mathrm{Im}\phi) &= (3,0) &&\Rightarrow \sqrt{3}\mathrm{Re}\dot{\phi} = (\mathrm{Re}\phi)^2 - \mathrm{Re}\phi &&\Leftrightarrow\ \sqrt{3}\dot{\phi} = \bar{\phi}^2 - \phi,\\
(\kappa,\mathrm{Im}\phi) &= (9,0) &&\Rightarrow \sqrt{3}\mathrm{Re}\dot{\phi} = (\mathrm{Re}\phi)^2 - 2\mathrm{Re}\phi &&\Leftrightarrow\ \sqrt{3}\dot{\phi} = \bar{\phi}^2 - 2\phi.
\end{aligned} \tag{7.70}$$

The (kink-type) solutions to (7.67) and (7.70) were given in [9]. They have zero energy and thus finite action only for $\kappa \in \{-3,-1,3\}$, with the two latter also displayed in (7.49) and (7.50), respectively. Additionally, finite action bounce solutions to (7.68) can numerically be constructed for $\kappa < -3$ and $3 < \kappa < 5$.

7.3 Instanton-Anti-Instanton Chains and Dyons

If we replace $\mathbb{R} \times G/H$ by $S^1 \times G/H$, the time interval will be of finite length, namely the circle circumference L, and we are after solutions periodic in τ. In this case, the action is always finite, and the $E = 0$ requirement gets replaced by

$$\phi_i(\tau + L) = \phi_i(\tau). \tag{7.71}$$

The physical interpretation of such configurations is one of instanton-anti-instanton chains.

7.3.1 Periodic Solutions. Consider the case of $\phi_1 = \phi_2 = \phi_3 =: \phi$ in (7.26), which means $G/H = G_2/SU(3)$. For the magical values of κ we get the following second-order equations, on which we must impose the periodicity condition (7.71):

$$\begin{aligned}
(\kappa, \mathrm{Re}\phi) &= (-1, -1/2) &&\Rightarrow \frac{3}{2}\mathrm{Im}\ddot{\phi} = \mathrm{Im}\phi\left(\mathrm{Im}\phi^2 - \frac{3}{4}\right), \\
(\kappa, \mathrm{Re}\phi) &= (-3, 0) &&\Rightarrow \frac{3}{2}\mathrm{Im}\ddot{\phi} = \mathrm{Im}\phi\left(\mathrm{Im}\phi^2 - 1\right), \\
(\kappa, \mathrm{Re}\phi) &= (-7, 1) &&\Rightarrow \frac{3}{2}\mathrm{Im}\ddot{\phi} = \mathrm{Im}\phi\left(\mathrm{Im}\phi^2 - 3\right), \\
(\kappa, \mathrm{Re}\phi) &= (-3, 0) &&\Rightarrow \frac{3}{2}\mathrm{Re}\ddot{\phi} = \mathrm{Re}\phi\left(\mathrm{Re}\phi - \frac{1}{2}\right)(\mathrm{Re}\phi - 1) \\
(\kappa, \mathrm{Re}\phi) &= (-9, 0) &&\Rightarrow \frac{3}{2}\mathrm{Re}\ddot{\phi} = \mathrm{Re}\phi\left(\mathrm{Re}\phi - 1\right)(\mathrm{Re}\phi - 2).
\end{aligned} \tag{7.72}$$

At finite L, we obtain a different kind of solution (sphalerons), namely

$$\phi(\tau) = \beta \pm \mathrm{i}\sqrt{3}\gamma\, k\, b(k)\, \mathrm{sn}[b(k)\gamma\tau; k], \tag{7.73}$$

for the values

$$(\kappa; \beta, \gamma) \in \left\{\left(-1; -\frac{1}{2}, 1\right), \left(-3; 0, \frac{2}{\sqrt{3}}\right), (-7; 1, 2)\right\} \tag{7.74}$$

and

$$\phi(\tau) = \beta \pm \sqrt{3}\gamma\, k\, b(k)\, \mathrm{sn}[b(k)\gamma\tau; k], \tag{7.75}$$

for

$$(\kappa; \beta, \gamma) \in \left\{\left(3; \frac{1}{2}, \frac{1}{\sqrt{3}}\right), \left(9; 1, \frac{2}{\sqrt{3}}\right)\right\}. \tag{7.76}$$

Here, $b(k) = (2 + 2k^2)^{-1/2}$ and $0 \le k \le 1$. Since the Jacobi elliptic function has a period of $4K(k)$ (see appendix E), the periodicity condition (7.71) is satisfied if

$$\gamma\, b(k)\, L = 4K(k)n, \qquad n \in \mathbb{N}, \tag{7.77}$$

which fixes $k = k(L, n)$, so that $\phi(\tau; k(L, n)) =: \phi^{(n)}(\tau)$. Solutions (7.73) and (7.75) exist if $L \geq 2\pi\sqrt{2}\, n$ (see e.g. [20]).

By virtue of the periodic boundary conditions (7.71), the topological charge of the sphaleron $\phi^{(n)}$ is zero. Such a configuration is interpreted as a chain of n kinks and n antikinks, alternating and equally spaced around the circle (see e.g. [20]). The energy of the sphaleron, interpreted as a static configuration on $S^1 \times G/H$, is

$$E = \int_0^L \mathrm{d}\tau \left\{ |\dot{\phi}|^2 + V(\phi, \bar{\phi}) \right\}, \tag{7.78}$$

so e.g. for the case $\kappa = -3$ in (7.75) we obtain

$$E(\phi_n) = \frac{2n}{3\sqrt{2}} \Big[8\big(1 + k^2\big) E(k) - \big(1 - k^2\big)\big(5 + 3k^2\big) K(k) \Big], \tag{7.79}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively.

The solutions (7.75) can be embedded into the other cosets G/H, where they are special solutions with $\phi_1 = \phi_2$ or $\phi_1 = \phi_2 = \phi_3$, respectively. By applying the symmetry transformation (7.61) or (7.30), respectively, their degeneracy may be lifted. Substituting the solutions back into the action (6.39), we obtain finite-action Yang-Mills configurations. These are interpreted as chains of n instanton-anti-instanton pairs sitting on $S^1 \times G/H$ with a six-dimensional nearly Kähler coset space G/H. Away from the magical κ values, such chains are to be found numerically.

7.3.2 Dyonic Solutions. Let us change the signature of the metric on $\mathbb{R} \times G/H$ from Euclidean to Lorentzian by choosing on $\mathbb{R}$ a coordinate $t = -\mathrm{i}\tau$, so that $\tilde{e}^0 = \mathrm{d}t = -\mathrm{i}\mathrm{d}\tau$. Then the metric on $\mathbb{R} \times G/H$ is given by

$$\mathrm{d}s^2 = -(\tilde{e}^0)^2 + \delta_{ab} e^a e^b. \tag{7.80}$$

For the cases discussed above, we arrive at the same second-order differential equations as in Euclidean space, except for the replacement

$$\ddot{\phi}_i \to -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi_i. \tag{7.81}$$

In particular, this implies a sign change of the left-hand side relative to the right-hand side in (7.26), (7.58) and (7.64). Thus, in the Lagrangian we effectively have a sign flip of the potential V, so that the analog Newtonian dynamics for $(\phi_i(t))$ is based on $+V$.

Again, consider the case $G_2/SU(3)$. Then the equations of motion are given by

$$6\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi = -(\kappa - 1)\phi + (\kappa + 3)\bar{\phi}^2 - 4|\phi|^2\phi = -\frac{1}{3}\frac{\partial V}{\partial \bar{\phi}}, \tag{7.82}$$

where V is given by (7.65). This equation can be integrated for the special values of $\kappa \in \{-1, -3, -7, 3, 9\}$, which yields

$$\phi(t) = \beta \pm \mathrm{i}\sqrt{\frac{3}{2}}\,\gamma \cosh^{-1} \frac{\gamma t}{\sqrt{2}} \tag{7.83}$$

for

$$(\kappa; \beta, \gamma) \in \left\{ \left(-1; -\frac{1}{2}, 1\right), \left(-3; 0, \frac{2}{\sqrt{3}}\right), (-7; 1, 2) \right\} \tag{7.84}$$

and

$$\phi(t) = \beta \pm \sqrt{\frac{3}{2}}\,\gamma \cosh^{-1} \frac{\gamma t}{\sqrt{2}} \tag{7.85}$$

for the values

$$(\kappa; \beta, \gamma) \in \left\{ \left(3; \frac{1}{2}, \frac{1}{\sqrt{3}}\right), \left(9; 1, \frac{2}{\sqrt{3}}\right) \right\}. \tag{7.86}$$

Via the 3-symmetry action, these solutions are mapped to rotated ones. Any such configuration is a bounce in our double-well-type potential, which most of the time hovers around a saddle point. For other values of κ, such bounce solutions may be found numerically.

Inserting (7.83) or (7.85) into the gauge potential, we arrive at dyon-type configurations with smooth nonvanishing 'electric' and 'magnetic' field strength $\mathcal{F}_{0a}$ and $\mathcal{F}_{ab}$, respectively. The total energy

$$-\mathrm{Tr}(2\,\mathcal{F}_{0a}\mathcal{F}_{0a} + \mathcal{F}_{ab}\mathcal{F}_{ab}) \times \mathrm{Vol}(G/H) \tag{7.87}$$

for these configurations is finite, but their action diverges unless $\phi(\pm\infty) = \mathrm{e}^{2\pi\mathrm{i}k/3}$. These are saddle points for $\kappa < -3$ and $\kappa > 4$. Thus, for $|\kappa - 1| > 4$, the potential (7.65) admits pairs $\phi_\pm(t)$ of finite-action dyons, with

$$\phi_\pm(\pm\infty) = 1 \quad \text{and} \quad \phi_\pm(0) = \frac{1}{6}\left(\kappa - 3 \pm \sqrt{\kappa^2 - 9}\right) \quad \text{for } \kappa > 5 \tag{7.88}$$

and a more complex behavior for $\kappa < -3$. The $\kappa = -7$ and $\kappa = 9$ straight line solutions in (7.83) and (7.85) are among these. Numerical trajectories for some intermediate values are shown in the plots of figure 7.4 and 7.5.

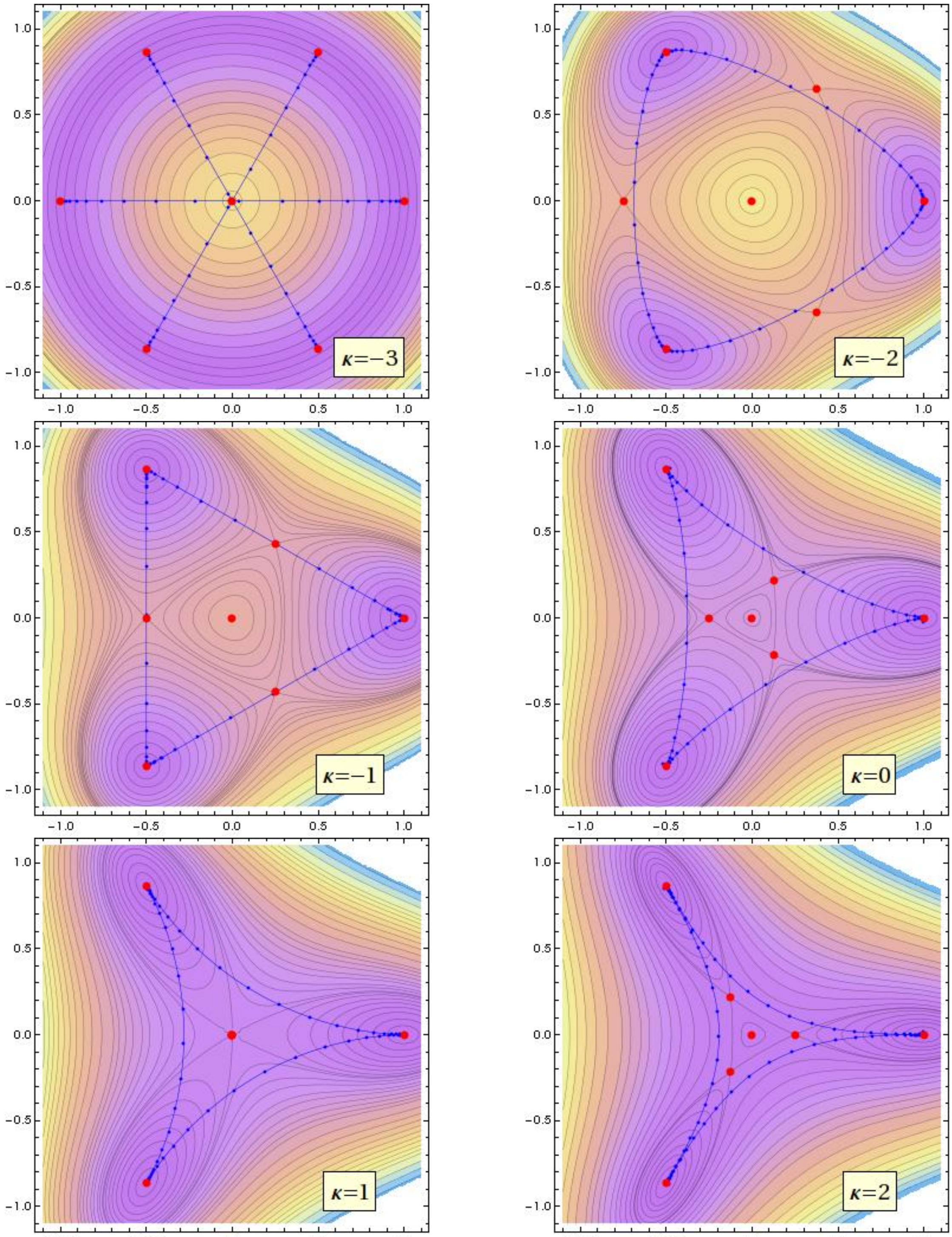

Figure 7.1: Numerical solutions for $\kappa \in \{-3; -2; -1; 0; 1; 2\}$.

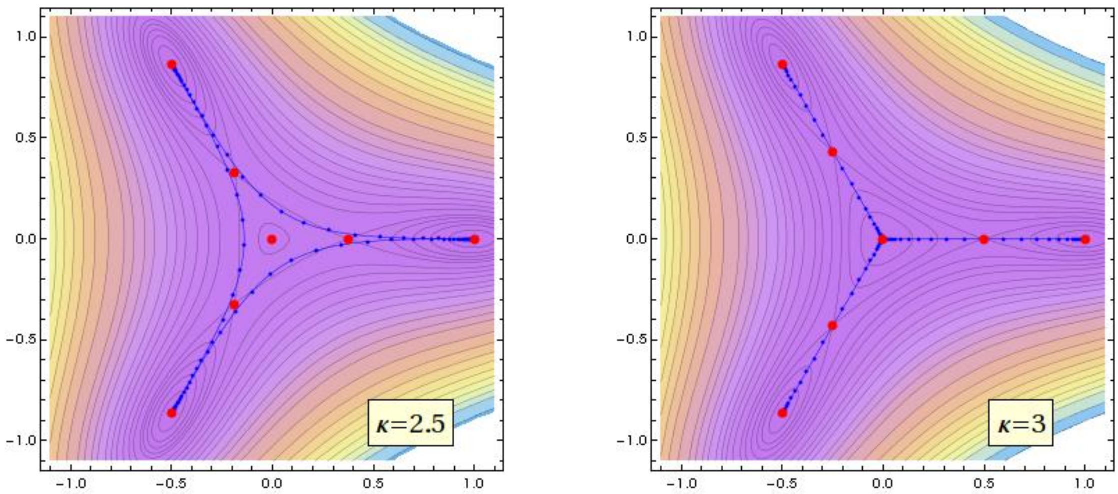

Figure 7.2: Numerical solutions for $\kappa \in \{2.5; 3\}$.

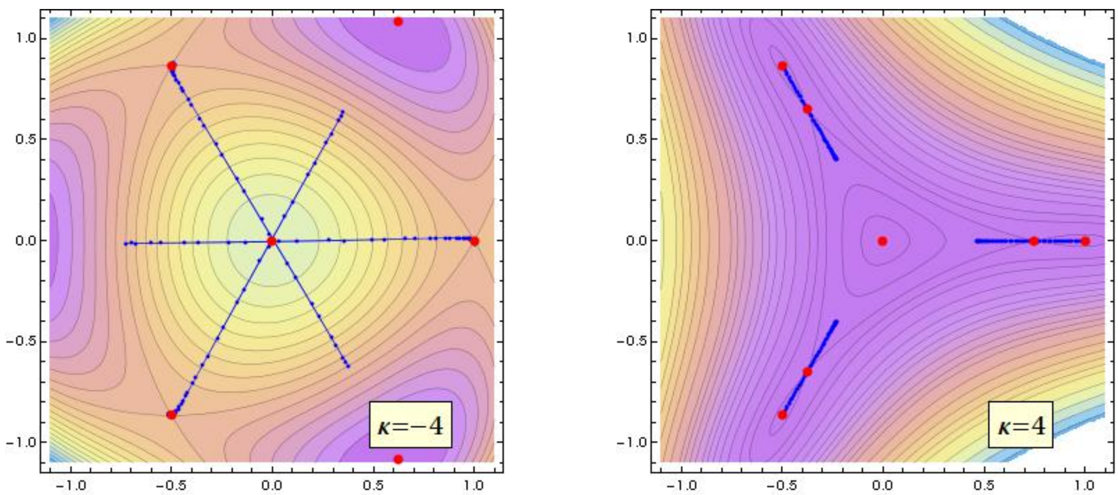

Figure 7.3: Numerical solutions for $\kappa \in \{-4; 4\}$.

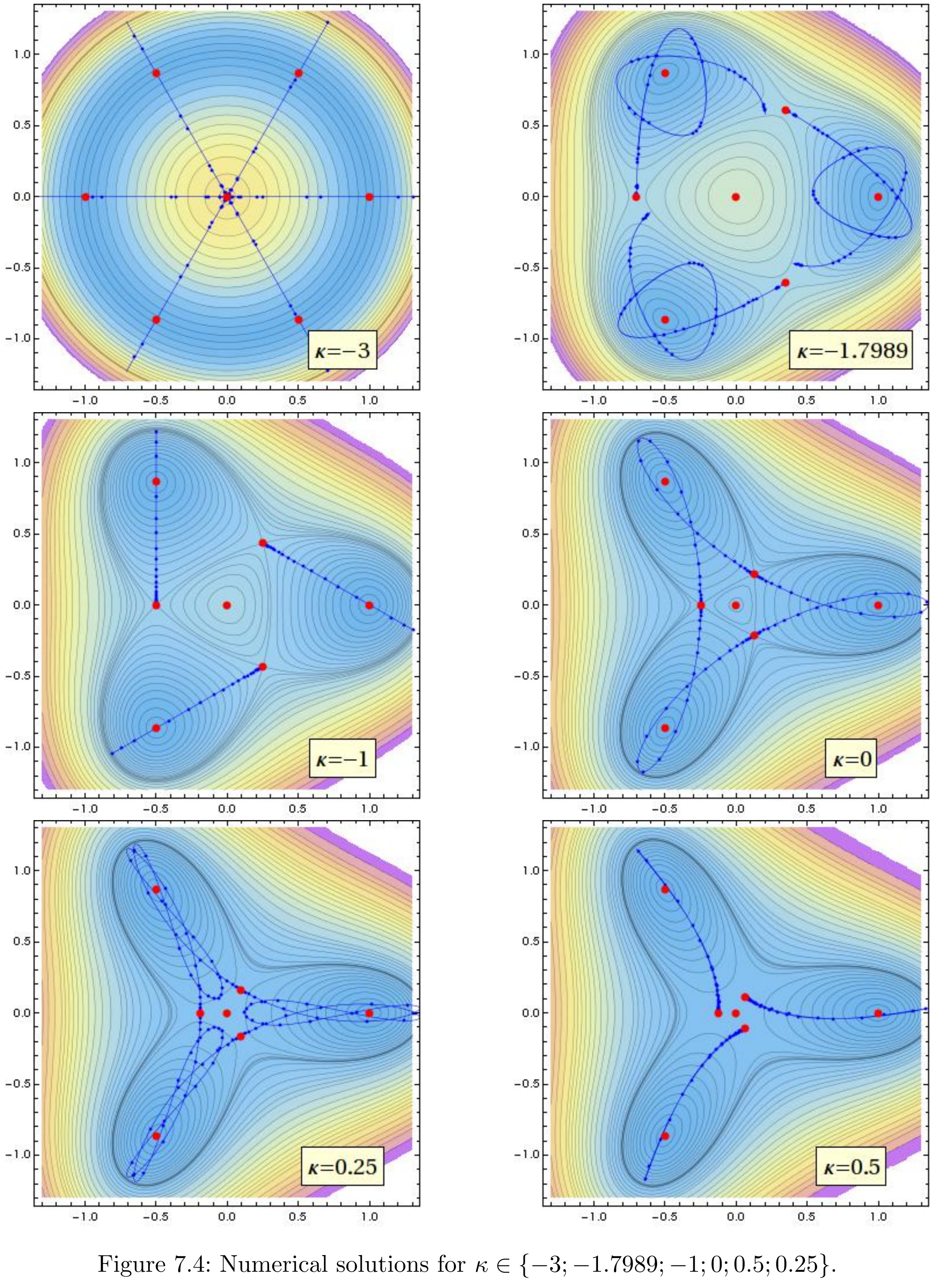

Figure 7.4: Numerical solutions for $\kappa \in \{-3; -1.7989; -1; 0; 0.5; 0.25\}$.

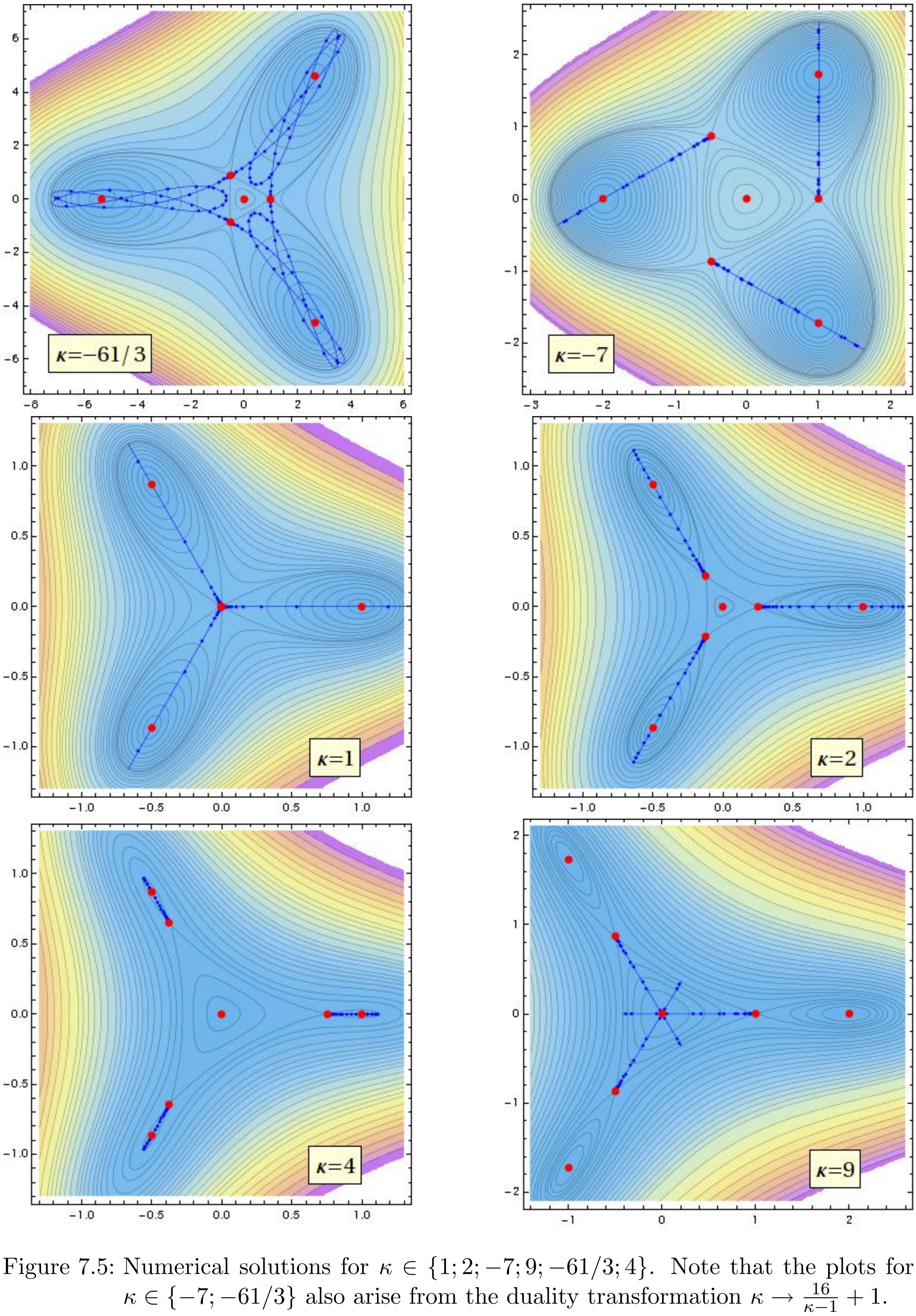

Figure 7.5: Numerical solutions for $\kappa \in \{1; 2; -7; 9; -61/3; 4\}$. Note that the plots for $\kappa \in \{-7; -61/3\}$ also arise from the duality transformation $\kappa \to \frac{16}{\kappa-1} + 1$.

Chapter 8

Conclusion

We considered spaces of the form $\mathcal{M} = \mathbb{R} \times G/H$, where G/H is a nearly Kähler manifold, so that $\mathcal{M}$ has structure group G_2. On these spaces, we considered the Yang-Mills equations with torsion, which for example arise in String Theory. We constructed some explicit (analytical) solutions to these equations, as well as numerical ones, where the analytical solutions are of kink or bounce type when choosing a real coordinate on the space $\mathbb{R}$ and of dyon type when choosing an imaginary coordinate.

These gauge-field solutions are instantons (for a real coordinate) and dyons (for an imaginary coordinate). In particular the instantons play an important rôle in the quantized theories, though classically obtained. This is because the classical solution survives the quantization process relatively unscathed. It provides a contribution to the path integral and is a good first approximation to the properties of the quantum theoretical solution. Further the solutions are inherently non-perturbative, i.e. they cannot be obtained by applying a perturbative expansion in some coupling constant. In the case of instantons, which are related to tunnelling effects between (stable) vacua, they give an approximation of the structure of the vacuum state of the quantum theory. The dyonic solutions are related to magnetic monopoles in these theories. [21, 40].

Considering the numerical kink-type solutions, it is interesting that with variation of the parameter κ the solutions seem to interpolate between the analytical ones at the special values $\kappa = -1$ and $\kappa = 3$. Since the numerical simulations were quite unstable (i.e. sensitive with respect to the initial values), we were only able to interpolate in an intervall of finite length. We therefore could not choose "zero velocity" at $t \to \pm\infty$ and there may be room for more investigation, despite the fact that the results (the interpolation between analytical solutions) seem quite plausible.

The numerical solutions for the dyonic case seem more stable, but here we had nearly no dependence on the starting angle of the trajectory. This means that varying the angle in a wide range produced almost the same trajectories. As one can see from the results, the solutions do not seem to interpolate between the analytical ones. In particular, for most values of κ we did not find "good" trajectories, but rather chaotic ones.

Outlook. A possible extension is to consider higher-dimensional spaces of the form $\mathcal{M} = \Sigma \times G/H$, with a 2-dimensional space Σ. Some work has already been done in this direction [11].

As a last remark, consider 10-dimensional heterotic String Theory. Now to compactify this theory, one is looking for subgroups of the gauge group E_8. Such subgroups may be constructed using the technique presented in [33], and the resulting space of the form $\mathcal{M}^4 \times X^6$ may be investigated.

Appendix A

Associative and Non-associative Algebras

Apart from the widely known number fields $\mathbb{R}$ and $\mathbb{C}$, there are some more types of numbers which are of interest in physics [3,5,7]. These are the algebra of the quaternions $\mathbb{H}$ and the (non-associative) algebra of the octonions $\mathbb{O}$. In this chapter we present a generic construction method and write down a more explicit definition of these two. This material may be found for example in [2, 4, 7], and some further treatment of related algebras in [1].

A.1 Division Algebras and the Cayley-Dickson Construction

Definition A.1.1 (Algebra)**.** An *algebra* A over a field $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a bilinear product

$$A \times A \to A, \qquad (x, y) \mapsto x \cdot y, \tag{A.1}$$

which we call *A-multiplication* or simply *multiplication*. If the meaning is clear, we may write $xy := x \cdot y$. If there is an element $1 \in A$, such that $1 \cdot x = x$ for all $x \in A$, we call A an *algebra with* 1 or *unital algebra*. If the multiplication is commutative, that is $x \cdot y = y \cdot x$ for all $x, y \in A$, we call A *commutative*, otherwise *non-commutative*. We call A *associative*, if the multiplication is associative, i.e $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in A$.

$A \neq \{0\}$ is a *division algebra*, if for any two elements $a, b \in A$, $b \neq 0$, there exist precisely one element $x \in A$ with $a = x \cdot b$ and precisely one element $y \in A$ with $a = b \cdot y$.

A *normed division algebra* A is a division algebra which is also a normed vector space over the real or complex numbers, with a norm $\| \cdot \|$ that satisfies

$$\|xy\| = \|x\| \cdot \|y\| \tag{A.2}$$

for all $x, y \in A$.

We now present a way to construct all normed division algebras over $\mathbb{R}$, known as the *Cayley-Dickson construction*.

A complex number $z := a + ib$ can be thought of a pair of real numbers $z = (a, b)$. Together with the obvious addition and the multiplication rule

$$(a, b)(c, d) := (ac - db, ad + cb) \tag{A.3}$$

they form the field $\mathbb{C}$. We define the *conjugation* of such a pair (a, b) as

$$(a, b)^* := (a, -b). \tag{A.4}$$

If we make the usual identification $(a, 0) \equiv a \in \mathbb{R}$, we may also rewrite equation (A.3),

$$(a, b)(c, d) = (ac - db^*, a^*d + cb), \tag{A.5}$$

as complex conjugation acts as the identity on $\mathbb{R}$. The same happens if we rewrite (A.4) as

$$(a, b)^* = (a^*, -b). \tag{A.6}$$

From the equations (A.5) and (A.6) we can construct new algebras from the previous ones. This way of getting new algebras is known as the *Cayley-Dickson construction.*

Applying this construction to $\mathbb{R}$, we successively get $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. But this does not lead to an infinite sequence of division algebras, since each time we apply the construction, we loose a property of the previous algebra: From $\mathbb{R}$ to $\mathbb{C}$ we loose the ordering property, from $\mathbb{C}$ to $\mathbb{H}$ we loose commutativity and from $\mathbb{H}$ to $\mathbb{O}$ we loose associativity. The next time we apply Cayley-Dickson, we loose the division algebra property.

Theorem A.1.1 (Hurwitz)**.** *The only normed division algebras over $\mathbb{R}$ are (up to isomorphism) the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.*

Given a real division algebra $\mathbb{F}$, we may embed the real numbers via the identification

$$\mathbb{R} \hookrightarrow \mathbb{F}, \qquad x \mapsto (x, 0, \ldots, 0), \tag{A.7}$$

so that we may consider $\mathbb{R} \subset \mathbb{F}$ as a subfield. In this sense we may write $z := (x, y) \in \mathbb{F}$ with $x \in \mathbb{R}$, and call x the *real part* and y the *imaginary part* of z, denoted

$$x = \mathrm{Re}(z) \qquad \text{and} \qquad y = \mathrm{Im}(z). \tag{A.8}$$

We define the *conjugate* of z as

$$\bar{z} := z^* := (\mathrm{Re}(z), -\mathrm{Im}(z)) \in \mathbb{F}. \tag{A.9}$$

Another way of obtaining $\mathbb{H}$ and $\mathbb{O}$ is presented in the following section.

A.2 Quaternions, Octonions and Derivation Algebras

Definition A.2.1 (Quaternions)**.** The noncommutative algebra $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ of the quaternions is a 4-dimensional division algebra over $\mathbb{R}$. The defining equations are

$$e_i e_j = -\delta_{ij} e_4 + \sum_{k=1}^{3} \varepsilon_{ijk} e_k, \quad e_a e_4 = e_4 e_a = e_a, \quad e_4^2 = e_4 \tag{A.10}$$

for $i, j = 1, 2, 3$. Here ε_{ijk} is the totally antisymmetric Levi-Civita symbol and equal to unity for (ijk) equal to (123), (231), (312).

Remark A.2.1. The algebra $\mathbb{H}$ is indeed noncommutative:

$$e_1e_2 = \varepsilon_{123}e_3 = e_3, \tag{A.11}$$

whereas

$$e_2e_1 = \varepsilon_{213}e_3 = -e_3. \tag{A.12}$$

Definition A.2.2 (Octonions)**.** The equations defining the alternative nonassociative algebra of the octonions $\mathbb{O}$ are

$$e_ae_b = -\delta_{ab} + \sum_{c=1}^{7} f_{abc}e_c, \quad e_ae_8 = e_8e_a = e_a, \quad e_8^2 = e_8, \tag{A.13}$$

where $a, b = 1, \ldots, 7$ are the basic octonionic units and e_8 is the unit element in $\mathbb{O}$. The *Cayley structure constants* f_{abc} are totally antisymmetric in (abc) and equal to unity for the seven combinations (123), (145), (167), (246), (275), (365), (374).

Remark A.2.2.
(i) Following the same argument as above, $\mathbb{O}$ is also noncommutative. But $\mathbb{O}$ is also nonassociative:

$$e_1(e_2e_4) = e_1f_{246}e_6 = e_1e_6 = f_{167}e_7 = e_7, \tag{A.14}$$

whereas

$$(e_1e_2)e_4 = f_{123}e_3e_4 = f_{345}e_5 = -e_5. \tag{A.15}$$

(ii) Every 3 cyclically ordered units $e_i, e_j, e_k \in \mathbb{O}$ generate isomorphic copies of the quaternion algebra in $\mathbb{O}$.

Definition A.2.3 (Derivation)**.** Let $\mathbb{F}$ be a field and A be an algebra over $\mathbb{F}$. A *derivation* is an $\mathbb{F}$-linear map $D : A \to A$, which satisfies the Leibniz rule

$$D(ab) = (Da)b + a(Db), \qquad \forall a, b \in A. \tag{A.16}$$

Remark A.2.3.
(i) From the $\mathbb{F}$-linearity and the Leibniz rule it follows that $D(x) = 0$, for all $x \in \mathbb{F}$.

(ii) As D is a linear map, it suffices to specify how it acts on some basis of the algebra A.

Definition A.2.4. Let A be an $\mathbb{F}$-algebra. Then we denote the collection of all derivations from A to itself by $\mathfrak{Der}(A)$.

Remark A.2.4.
(i) We consider the derivation algebra of the octonions, $\mathfrak{Der}(\mathbb{O})$. From (A.13) it follows that for every element $D \in \mathfrak{Der}(\mathbb{O})$ and $a, b = 1, \ldots, 7$ we have

$$D(e_ae_b) = \sum_{c=1}^{7} f_{abc}D(e_c), \qquad D(e_8e_a) = D(e_ae_8) = D(e_a), \qquad D(e_8) = 0. \tag{A.17}$$

(ii) Since a derivation is by definition a linear map, we may represent all possible derivations on an 8-dimensional space by 8×8 matrices, which gives in general 64 parameters which we are free to choose. Since $D(e_8) = 0$, we fix one column of the matrices, leaving 56 parameters. From the definition of octonions, we see that we have 42 non-vanishing structure coefficients which define the octonionic multiplication. Since these relations have to be conserved under a linear map, we have at least $56-42 = 14$ parameters free. Now one can show that $\dim \mathfrak{Der}(\mathbb{O}) = 14$. The algebra of derivations of the octonions is denoted by

$$\mathfrak{g}_2 := \mathfrak{Der}(\mathbb{O}). \tag{A.18}$$

Appendix B

Lie Algebras

In the process of describing the symmetries of physical systems, the mathematical concept of a group naturally appears. A particular type of groups are the so-called *Lie groups*, which additionally have the structure of a differentiable manifold. When describing these groups, which may be geometrically quite complicated, one arrives at the *Lie algebras* of the groups. They are vector spaces (and therefore geometrically simpler), but reflect some properties of the group they are derived from. The literature for this chapter is [8, 45–47, 51, 53, 54]. In particular [51] gives a deeper treatment with a similar aim as this thesis.

B.1 Groups, Lie Groups and Lie Algebras

Definition B.1.1 (Topological Group)**.** A *topological group* G is a topological space, such that the group operation,

$$G \times G \to G, \quad (x, y) \mapsto xy,$$

and taking inverses,

$$G \to G, \quad x \mapsto x^{-1}$$

are continuous functions.

Definition B.1.2 (Group Action)**.** Let G be a group with identity element $e \in G$ and M be a set. A (left) *operation* or a (left) *action* of G onto M is a map

$$\circ : G \times M \to M, \qquad (g, x) \mapsto g \circ x, \tag{B.1}$$

such that

(i) $e \circ x = x$, for all $x \in M$.

(ii) $g \circ (h \circ x) = (g \circ h) \circ x$, for all $g, h \in G$ and for all $x \in M$.

Definition B.1.3 (Isotropy Group, Orbit)**.** Let G be a group acting on a set M. We define the *isotropy group* (or *stabilizer group*) G_x of $x \in M$ in G as

$$G_x := \{g \in G : gx = x\}. \tag{B.2}$$

G_x is a subgroup of G.

Let $x \in M$. The subset M_g in M consisting of all elements gx, denoted by Gx, is called the *orbit* of x under G.

Definition B.1.4. Let G be a group acting on a set M. Let $K := \bigcap_{x \in M} G_x$. An action or operation of G onto M is said to be *faithful*, if $K = \{e\}$. A *fixed point* of G is an element $x \in M$ such that $gx = x$ for all $g \in G$ or in other words $G = G_x$.

An action is said to be *transitive*, if there is only one orbit.

Example B.1.1. (i) *Conjugation.* For each $g \in G$, let $\gamma_g : G \to G$ be the map such that

$$\gamma_g(h) = ghg^{-1}. \tag{B.3}$$

Then $g \mapsto \gamma_g$ is a homomorphism $G \to \mathrm{Aut}(G)$, and so this map gives an operation of G on itself, called *conjugation*. The kernel of this homomorphism is a normal subgroup of G, which consists of all $g \in G$ such that $ghg^{-1} = h$, for all $h \in G$, i.e. all $g \in G$ which commute with every element of G. This kernel is called the *center* of G.

If A and B are two subsets of G, we say that they are *conjugate* if there exists $g \in G$ such that $B = gAg^{-1}$.

(ii) *Translation.* For each $g \in G$ we define the translation $T_g : G \to G$ by $T_g(h) := gh$. Then the map

$$(g, h) \mapsto gh = T_g(h) \tag{B.4}$$

defines an operation of G on itself.

Definition B.1.5 (Lie Group)**.** A *Lie group* is an algebraic group that has the structure of a differentiable manifold, such that the map

$$G \times G \to G, \qquad (g, h) \mapsto gh^{-1} \tag{B.5}$$

is smooth.

Example B.1.2. (i) $(\mathbb{R}^n, +)$ is a Lie group.

(ii) $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ with the complex multiplication and the manifold structure induced from $\mathbb{R}^2$.

(iii) $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ with matrix multiplication, regarded as open submanifolds of $\mathbb{R}^{n^2}$ resp. $\mathbb{R}^{2n^2}$.

(iv) Let G, H be Lie groups. Then their direct product $G \times H$, together with the direct product of their differentiable structures, is a Lie group.

Definition B.1.6 (Lie Subgroup)**.** Let G be a Lie group. A submanifold H of G is called a *Lie subgroup* if

(i) H is a subgroup of the (abstract) group G,

(ii) H is a topological group.

A Lie subgroup is itself a Lie group. To see this, consider the analytic mapping $\alpha : (x, y) \to xy^{-1}$ of $G \times G$ into G. Let α_H denote the restriction of α to $H \times H$. Then the mapping $\alpha_H : H \times H \to G$ is analytic and by (ii) the mapping $\alpha_H : H \times H \to H$ is continuous.

Definition B.1.7 (Lie Algebra)**.** A *Lie algebra* over a field $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ endowed with a bilinear map, the *Lie bracket*, denoted $(X, Y) \to [X, Y]$, for $X, Y \in \mathfrak{g}$, that satisfies $[X, Y] = -[Y, X]$ and the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{B.6}$$

If $F = \mathbb{R}$ or $F = \mathbb{C}$, we call $\mathfrak{g}$ a *real* resp. *complex Lie algebra.*

Example B.1.3. (i) $\mathbb{R}^n$ with the trivial Lie bracket $[\cdot, \cdot] \equiv 0$.

(ii) $\mathbb{R}^3$ with the vector product $[v, w] := v \times w$.

(iii) Every associative algebra with $[a, b] := ab - ba$.

(iv) Let $\mathbb{F}$ be a field. We consider an algebra A over $\mathbb{F}$ and its derivation algebra $\mathfrak{Der}(A)$, as defined in chapter A.2. If we define the Lie bracket by $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ for $D_1, D_2 \in \mathfrak{Der}(A)$, we note that it is $\mathbb{F}$-linear and satisfies the Leibnitz rule,

$$[D_1, D_2](ab) = a([D_1, D_2]b) + ([D_1, D_2]a)b, \tag{B.7}$$

so every derivation algebra is a Lie algebra. Especially the algebra $\mathfrak{g}_2$, as defined in (A.18), is a Lie algebra.

Definition B.1.8. A *subalgebra* of a real or complex Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$, such that $[H_1, H_2] \in \mathfrak{h}$ for all $H_1, H_2 \in \mathfrak{h}$. If $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{h}$ is a real subspace of $\mathfrak{g}$ which is closed under brackets, then $\mathfrak{h}$ is said to be a *real subalgebra* of $\mathfrak{g}$.

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a *Lie algebra homomorphism* if $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. If, in addition, ϕ is one-to-one and onto, then ϕ is called a *Lie algebra isomorphism.* A Lie algebra isomorphism of a Lie algebra with itself is called a *Lie algebra automorphism.*

A subalgebra of a Lie algebra is, again, a Lie algebra. A real subalgebra of a complex Lie algebra is a real Lie algebra. The inverse of a Lie algebra isomorphism is, again, a Lie algebra isomorphism.

Proposition B.1.1. *Let $\mathfrak{g}$ be a finite-dimensional real or complex Lie algebra and $I_1, \ldots, I_n$ be a basis for $\mathfrak{g}$ (as a vector space). Then for each i, j, $[I_i, I_j]$ can be written uniquely in the form*

$$[I_i, I_j] = \sum_{k=1}^{n} f_{ijk} I_k, \tag{B.8}$$

with constants f_{ijk}. These constants satisfy

$$f_{ijk} + f_{jik} = 0, \tag{B.9}$$

$$\sum_{m=1}^{n} (f_{ijm} f_{mkl} + f_{jkm} f_{mil} + f_{kim} f_{mjl}) = 0. \tag{B.10}$$

Proof. Since $\{I_k\}_k$ form a basis of $\mathfrak{g}$ and since $[I_i, I_j] \in \mathfrak{g}$ by definition, the proof follows. □

Definition B.1.9. The f_{ijk} from proposition B.1.1 are called *structure constants* or *structure coefficients* of the Lie algebra $\mathfrak{g}$ (with respect to the chosen basis). The I_i's are sometimes called the *(infinitesimal) generators* of the corresponding Lie group.

Our goal is to associate a finite-dimensional, real Lie algebra to every finite dimensional Lie group. Therefore, we introduce the following notions. For every fixed element $g \in G$ we have the diffeomorphisms

$$L_g : x \in G \to g \cdot x \in G \qquad \text{(left translation)},$$
$$R_g : x \in G \to x \cdot g \in G \qquad \text{(right translation)}$$

and

$$\alpha_g := L_g \circ R_g^{-1} : G \to G \qquad \text{(inner automorphism)}.$$

Let M be a manifold, $F : M \to M$ a diffeomorphism and $X \in \mathfrak{X}(M)$ a vector field on M. Then we define $dF(X)$ as the vector field given by

$$dF(X)(x) := dF_{F^{-1}(x)} X(F^{-1}(x)), \quad x \in M.$$

One can show that this definition is compatible with the Lie bracket:

$$dF([X, Y]) = [dF(X), dF(Y)], \quad \forall X, Y \in \mathfrak{X}(M). \tag{B.11}$$

We call a vector field $X \in \mathfrak{X}(G)$ on a Lie group G *left invariant* resp. *right invariant*, if $dL_g(X) = X$ resp. $dR_g(X) = X$ holds for all $g \in G$. Because of (B.11) the vector space

$$\mathfrak{g} := \{X \in \mathfrak{X}(G) : X \text{ is left invariant}\}$$

together with the vector field commutator is a Lie algebra.

Definition B.1.10. The Lie algebra of left invariant vector fields $(\mathfrak{g}, [\cdot, \cdot])$ is called the *Lie algebra of the Lie group G*. If the group is denoted G, we write $\mathfrak{g}$ or $\mathrm{Lie}(G)$ for its algebra.

Remark B.1.1. Every left-invariant vector field $X \in \mathfrak{g}$ is uniquely determined by the vector $X(e) \in T_eG$, the tangent space of the identity $e \in G$, because

$$X(g) = dL_g(X(e)) \quad \text{for } g \in G. \tag{B.12}$$

We will therefore identify $\mathfrak{g}$ and T_eG.

Example B.1.4. (i) Let $G = GL(n, \mathbb{R})$ the Lie group of all invertible real $(n \times n)$-matrices. The corresponding Lie algebra is the vector space $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ of all real $(n \times n)$-matrices with the commutator

$$[X, Y] = X \circ Y - Y \circ X, \quad X, Y \in \mathfrak{g}.$$

This is seen as follows: Since $GL(n, \mathbb{R})$ is an (open) submanifold of $\mathbb{R}^{n^2}$, one can identify the tangent space $T_I G$ at the identity matrix I with $\mathbb{R}^{n^2} = \mathfrak{gl}(n, \mathbb{R})$.

We calculate the commutator. Let $X \in T_I G$ and γ_X be a smooth curve in $G = GL(n, \mathbb{R})$ with $\gamma_X(0) = I$ and $\gamma_X'(0) = X$. The left-invariant vector field $\widetilde{X}$ generated by X then satisfies by (B.12) and by definition of the directional derivative

$$\begin{aligned} \widetilde{X} &= dL_A(X) = \frac{d}{dt} \left(L_A(\gamma_X(t))\right)_{|t=0} \\ &= \frac{d}{dt} \left(A \circ \gamma_X(t)\right)_{|t=0} = A \circ X. \end{aligned} \tag{B.13}$$

Now the commutator on vector fields can be calculated in terms of directional derivatives:

$$\begin{aligned} [X, Y] &= [\widetilde{X}, \widetilde{Y}](I) = \widetilde{X}(\widetilde{Y})(I) - \widetilde{Y}(\widetilde{X})(I) \\ &= \frac{d}{dt} \left(\widetilde{Y}(\gamma_X(t))\right)_{|t=0} - \frac{d}{dt} \left(\widetilde{X}(\gamma_Y(t))\right)_{|t=0} \\ &\overset{(B.13)}{=} \frac{d}{dt} \left(\gamma_X(t) \circ Y - \gamma_Y(t) \circ X\right)_{|t=0} \\ &= X \circ Y - Y \circ X. \end{aligned}$$

(ii) Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$, $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be two Lie algebras. The direct sum $\mathfrak{g} \oplus \mathfrak{h}$ with the commutator

$$[X_1 + Y_1, X_2 + Y_2] := [X_1, X_2]_{\mathfrak{g}} + [Y_1, Y_2]_{\mathfrak{h}}, \quad X_1, X_2, Y_1, Y_2 \in \mathfrak{h}$$

then again is a Lie algebra.

Let G, H be two Lie groups with corresponding Lie algebras $\mathfrak{g}, \mathfrak{h}$. The Lie algebra of $G \times H$ then is $\mathfrak{g} \oplus \mathfrak{h}$.

We note

Theorem B.1.1. *Let G be a Lie group. If H is a Lie subgroup of G, then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$, the Lie algebra of G. Each subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one connected Lie subgroup of G.*

B.2 The Exponential Map

Definition B.2.1. A *one-parameter subgroup* of a Lie group G is a continuous homomorphism $\gamma : (\mathbb{R}, +) \to G$, i.e.

$$\gamma(t+s) = \gamma(t)\gamma(s), \quad t, s \in \mathbb{R}. \tag{B.14}$$

Since every element in G is invertible according to its group structure and $\gamma(0+0) = \gamma(0)\gamma(0)$, it follows that $\gamma(0) = e$.

We consider the one-parameter subgroup $\gamma_X(t)$ generated by the element $X \in \mathfrak{g} = T_eG$. It satisfies the following equations:

$$\gamma_X(0) = e, \quad \gamma_X(t+s) = \gamma_X(t)\gamma_X(s), \quad \dot{\gamma}_X(0) = X, \quad \forall s, t \in \mathbb{R}.$$

Definition B.2.2. We define the *exponential map* by

$$\exp : \mathfrak{g} \to G, \quad X \mapsto \exp X \equiv \mathrm{e}^X := \gamma_X(1). \tag{B.15}$$

With the help of the exponential map we now may introduce new coordinates on the Lie group. To do this, we note:

Theorem B.2.1. *Let* $\mathfrak{g}$ *be the Lie algebra of a Lie group* G *and* $\exp : \mathfrak{g} \to G$ *the exponential map. Then* $\exp$ *is smooth and it is a local diffeomorphism around* $0 \in \mathfrak{g}$.

Proof. The smoothness follows directly from the results on ordinary differential equations and their smooth depedence on initial values (Picard-Lindelöf theorem). We show that

$$\mathrm{d}\exp_0 : T_0\mathfrak{g} \cong \mathfrak{g} \to T_eG \cong \mathfrak{g}$$

is an isomorphism, i.e. that exp is a local diffeomorphism around $0 \in \mathfrak{g}$. The identification $T_0\mathfrak{g}$ with $\mathfrak{g}$ is valid because $\mathfrak{g}$ is a finite-dimensional vector space and so its tangent space at 0 is also finite dimensional with the same dimension. We choose a left-invariant vector field $X \in \mathfrak{g}$. Then

$$\mathrm{d}\exp_0(X) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(0 + tX)\right)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp tX\right)_{|t=0} = X(e).$$

By the identification $T_eG = \mathfrak{g}$ we get $\mathrm{d}\exp_0 = \mathrm{id}_{\mathfrak{g}}$. $\square$

Definition B.2.3. Let M be a manifold, G be a Lie group with a left action defined on M and $X \in \mathfrak{g}$. We call the vector field $\widetilde{X}$, defined by

$$\widetilde{X}(p) := \frac{\mathrm{d}}{\mathrm{d}t}(\exp(-tX) \cdot p)_{|t=0} \tag{B.16}$$

the *fundamental vector field* associated to X. If we have a right action defined on M, then we define the fundamental vector field by

$$\widetilde{X}(p) := \frac{\mathrm{d}}{\mathrm{d}t}(p \cdot \exp(tX))_{|t=0} \tag{B.17}$$

B.3 Adjoint Representation

The adjoint representation is a special homomorphism of a Lie group into the group $GL(\mathfrak{g})$ of all invertible linear maps of the Lie algebra $\mathfrak{g}$ of G.

Definition B.3.1. Let G be a Lie group and $(\mathfrak{h}, [\cdot,\cdot])$ a Lie algebra and V a vector space over $\mathbb{R}$ resp. $\mathbb{C}$. A Lie group homomorphism $\rho : G \to GL(V)$ is called a *representation* of the Lie group G over V. A *representation* of the Lie algebra $\mathfrak{h}$ over V is a Lie algebra homomorphism $\phi : \mathfrak{h} \to \mathfrak{gl}(V) = \mathrm{Lie}(GL(V))$, i.e. a linear map, which satisfies

$$\phi([X,Y]) = \phi(X) \circ \phi(Y) - \phi(Y) \circ \phi(X), \quad X, Y \in \mathfrak{h}.$$

The vector space V then is called the *representation space*.

We recall that a homomorphism between two Lie groups G and H is a $\mathscr{C}^\infty$-map $\rho : G \to H$ such that

$$\rho(gh) = \rho(g) \cdot \rho(h) \tag{B.18}$$

for all $g, h \in G$. If we denote the multiplication by some $g \in G$ by $m_g : G \to G$ and the multiplication by some $h \in H$ on H by $\widetilde{m}_h$, then m_g and $\widetilde{m}_h$ are differentiable maps (cf. definition B.1.5) and a $\mathscr{C}^\infty$-map $\rho : G \to H$ will be a homomorphism if it carries m_g to $\widetilde{m}_{\rho(g)}$ in the sense that the diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "m_g"'] & H \arrow[d, "\widetilde{m}_{\rho(g)}"] \\ G \arrow[r, "\rho"'] & H \end{tikzcd}$$

commutes.

Since the maps m_g have no fixed points, it is hard to associate to them any operation on the tangent space to G at one point. This suggests looking not at diffeomorphisms m_g, but at the automorphisms of G given by conjugation (cf. example B.1.1). For $g \in G$, we define the map

$$\Psi_g : G \to G, \qquad h \mapsto ghg^{-1}. \tag{B.19}$$

Now a homomorphism $\rho : G \to H$ respects the action of the group G on itself by conjugation, i.e. it will carry Ψ_g into $\widetilde{\Psi}_{\rho(g)}$ in the sense that the diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\Psi_g"'] & H \arrow[d, "\widetilde{\Psi}_{\rho(g)}"] \\ G \arrow[r, "\rho"'] & H \end{tikzcd}$$

commutes. That means we have a natural map

$$\Psi : G \to \mathrm{Aut}(G). \tag{B.20}$$

This map fixes the identity element $e \in G$. We can therefore extract some of its structure by looking at its differential at e.

Definition B.3.2. We set

$$\mathrm{Ad}\,(g) := (\mathrm{d}\Psi_g)_e : T_eG \to T_eG. \tag{B.21}$$

This is a representation

$$\mathrm{Ad}\, : G \to \mathrm{Aut}(T_eG) \tag{B.22}$$

of the group G on its own tangent space, called the *adjoint representation* of the group.

A homomorphism ρ respects the adjoint action of a group G on its tangent space T_eG at the identity. This means, for any $g \in G$ the actions of $\mathrm{Ad}\,(g)$ on T_eG and $\mathrm{Ad}\,(\rho(g))$ on T_eH must commute with the differential $(\mathrm{d}\rho)_e : T_eG \to T_eH$, i.e. the diagram

$$\begin{tikzcd} T_eG \arrow[r, "(\mathrm{d}\rho)_e"] \arrow[d, "\mathrm{Ad}\,(g)"'] & T_eH \arrow[d, "\mathrm{Ad}\,(\rho(g))"] \\ T_eG \arrow[r, "(\mathrm{d}\rho)_e"'] & T_eH \end{tikzcd}$$

commutes, or, equivalently

$$\mathrm{d}\rho(\mathrm{Ad}\,(g)(v)) = \mathrm{Ad}\,(\rho(g))(\mathrm{d}\rho(v)), \qquad \forall v \in T_eG. \tag{B.23}$$

We now want to get a condition that only depends on the differential $(\mathrm{d}\rho)_e$, i.e. we want to get a condition like (B.23), but without the homomorphism ρ. To do this, we take the differential of Ad .

Definition B.3.3. Since $\mathrm{Aut}(T_eG)$ is an open subset of $\mathrm{End}(T_eG)$, we identify its tangent space at the identity with $\mathrm{End}(T_eG)$. We then define the map

$$\mathrm{ad}\, : T_eG \to \mathrm{End}(T_eG), \qquad \mathrm{ad}\,(x) := \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}\,(\mathrm{e}^{tx})_{|t=0}. \tag{B.24}$$

Another way to view this is the following. The map ad generates an endomorphism for each vector $x \in T_eG$, i.e. $\mathrm{ad}\,(x) : T_eG \to T_eG$, so that ad may be viewed as a map

$$\mathrm{ad}\, : T_eG \times T_eG \to T_eG. \tag{B.25}$$

We use the notation $[\cdot,\cdot]$ for this bilinear map (cf. definition B.1.7), i.e. for a pair of tangent vectors $X, Y \in T_eG$ we write

$$[X, Y] = \mathrm{ad}\,(X)(Y). \tag{B.26}$$

We show, that this definition is equivalent to the previous one for the case of matrix groups.

Example B.3.1. We consider the Lie group $G := GL(n, \mathbb{R})$. Then the adjoint representation is given by

$$\mathrm{Ad}\,(g)(X) = gXg^{-1}, \tag{B.27}$$

which also extends to the ambient space $T_eG = \mathrm{End}(\mathbb{R}^n) \supset GL(n, \mathbb{R})$. For any pair of vectors $X, Y \in T_eG$ we consider the arc $\gamma : I \to G$ with $\gamma(0) = e$ and $\gamma'(0) = X$. We calculate

$$\begin{aligned}
[X, Y] = \mathrm{ad}\,(X)(Y) &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}\,(\gamma(t))(Y))_{|t=0} \\
&= \gamma'(0) \cdot Y \cdot \gamma(0) + \gamma(0) \cdot Y \cdot (-\gamma(0)^{-1} \cdot \gamma'(0) \cdot \gamma(0)^{-1}) \\
&= X \cdot Y - Y \cdot X.
\end{aligned}$$

Definition B.3.4. A Lie algebra $\mathfrak{g}$ is called *nilpotent*, if

$$\mathrm{ad}\,(x_1)\mathrm{ad}\,(x_2) \cdots \mathrm{ad}\,(x_k) = 0, \tag{B.28}$$

for any sequence x_i of elements of $\mathfrak{g}$ of sufficiently large length k.

Definition B.3.5. Let $\mathfrak{g}$ be a Lie algebra. A nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called *Cartan subalgebra*, if it is self-normalising (i.e. if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$).

If the underlying field $\mathbb{F}$ of the Lie algebra is algebraically closed of characteristic 0 and the Lie algebra is finite-dimensional, then all Cartan subalgebras are conjugate unter automorphisms of the Lie algebra and in particular are all isomorphic.

Example B.3.2. (i) Any nilpotent Lie algebra is its own Cartan subalgebra.

(ii) A Cartan subalgebra of the Lie algebra of $n \times n$ matrices over a field is the algebra of all diagonal matrices.

B.4 Matrix Lie Groups and their Algebras

As mentioned in example B.1.2, the general linear groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ of invertible matrices with entries in $\mathbb{R}$ or $\mathbb{C}$, respectively, are Lie groups.

We now consider closed subgroups G of $GL(n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. This means, if we have a converging sequence of matrices $A_n \in G$, the limit $A := \lim_{n\to\infty} A_n$ is either again in G or A is not invertible. Comparing this with definition B.1.6, we see that G is a topological group (since it is closed) and as such is a Lie subgroup. If we now consider topological subgroups G of $GL(n, \mathbb{F})$, we see that the limit of every converging sequence is either in G or not invertible. So we have found another characterization of Lie subgroups of the general linear group:

Lemma B.4.1. *Let G be a closed subgroup of $GL(n, \mathbb{R})$ resp. $GL(n, \mathbb{C})$. Then G is a Lie subgroup.*

Example B.4.1. (i) *The special linear groups $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$.* The special linear group (of $\mathbb{R}$ or $\mathbb{C}$) is the group of $n \times n$ invertible matrices (with real or complex entries) having determinant one. If now A_n is a sequence of matrices with determinant one and A_n converges to A, then A also has determinant one, because the determinant is a coninuous function. Thus $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ are matrix Lie (sub-)groups.

(ii) *The unitary and special unitary groups $U(n)$ and $SU(n)$.* An $n\times n$ complex matrix A is said to be *unitary*, if the column vectors of A are orthonormal, i.e. $A\bar{A}^T = \mathbf{1}$. $U(n)$ now is the set of all unitary matrices, whereas $SU(n)$ denotes the set of all unitary matrices with determinant one. In particular this means $|\det A| = 1$, which means that every (special) unitary matrix is invertible, so that $U(n)$ and $SU(n)$ are Lie subgroups of $GL(n,\mathbb{C})$.

(iii) *The symplectic groups $Sp(n,\mathbb{R})$, $Sp(n,\mathbb{C})$ and $Sp(n)$.* We define

$$Sp(n,\mathbb{F}) := \{A \in Mat(2n,\mathbb{F}) : A^T J A = J\} \quad \text{with} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{B.29}$$

I the identity in $GL(n,\mathbb{F})$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We further define the *compact symplectic group* as

$$Sp(n) := Sp(n,\mathbb{C}) \cap U(2n). \tag{B.30}$$

The group $Sp(n)$ may be also be regarded as a subgroup of the invertible quaternionic matrices $GL(n,\mathbb{H})$ which preserve the standard hermetian form on $\mathbb{H}^n$,

$$\langle x, y\rangle := \bar{x}^1 y^1 + \ldots + \bar{x}^n y^n. \tag{B.31}$$

Then the group of unit quaternions is given by

$$Sp(1) = \{q \in \mathbb{H} : |q| = 1\}. \tag{B.32}$$

Definition B.4.1. A matrix Lie group G is said to be *connected* if given any two matrices A and B in G, there exists a continuous path $A(t)$, $a \le t \le b$, lying in G with $A(a) = A$ and $A(b) = B$.

A matrix Lie group G which is not connected can be decomposed (uniquely) as a union of several pieces, called *components*, such that two elements of the same component can be joined by a continuous path, but two elements of different components cannot.

Proposition B.4.1. *If G is a matrix Lie group, then the component of G containing the identity is a subgroup of G.*

Proof. Saying A and B are both in the component containing the identity means that there exists continuous paths $A(t)$ and $B(t)$ with $A(0) = B(0) = I$, $A(1) = A$ and $B(1) = B$. Then $A(t)B(t)$ is a continuous path starting at I and ending at AB. Thus, the product of two elements of the identity component is again in the identity component. Furthermore, $A(t)^{-1}$ is a continuous path starting at I and ending at A^{-1}, and so the inverse of any element of the identity component is again in the identity component. Thus, the identity component is a subgroup. $\square$

Definition B.4.2. A non-abelian Lie algebra $\mathfrak{g}$ is called *simple*, if its only ideals[1] are 0 and $\mathfrak{g}$. A direct sum of simple Lie algebras is called a semisimple Lie algebra.

Proposition B.4.2. *The Lie algebra $\mathfrak{g}$ of a matrix Lie group G is a real Lie algebra.*

Proof. $\mathfrak{g}$ is a real subalgebra of the space $Mat(n, \mathbb{C})$ of all complex matrices and is, thus, a real Lie algebra. $\square$

We mention the following theorem, which justifies the restriction to matrix Lie algebras.

Theorem B.4.1 (Ado)**.** *Every finite-dimensional real Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. Every finite-dimensional complex Lie algebra is isomorphic to a complex subalgebra of $\mathfrak{gl}(n, \mathbb{C})$.*

We now want to construct Lie algebras for some matrix Lie groups. Considering definition B.1.10, remark B.1.1 and theorem B.2.1, we see that for a matrix Lie group G its Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g} := \{X \in Mat(n, \mathbb{F}) : \exp(tX) \in G \; \forall t \in \mathbb{R}\}, \tag{B.33}$$

with the commutator given by B.3.1. On the other hand we may obtain the group elements which are connected to the identity by differentiating $\exp(tX)$ for $X \in \mathfrak{g}$ and setting $t = 0$.

Example B.4.2. (i) *The general linear groups.* If X is any $n \times n$ complex matrix, then $\exp(tX)$ is invertible. Thus

$$\mathfrak{gl}(n, \mathbb{C}) = Mat(n, \mathbb{C}). \tag{B.34}$$

If X is real, then $\exp(tX)$ is also real, so that

$$\mathfrak{gl}(n, \mathbb{R}) = Mat(n, \mathbb{R}). \tag{B.35}$$

(ii) *The unitary groups.* A matrix U is unitary if and only if $\bar{U}^T = U^{-1}$. Thus $\exp(tX)$ is unitary if and only if $\exp(t\bar{X}^T) = \exp(-tX)$, i.e. differentiating at $t = 0$ yields the condition

$$\mathfrak{u}(n) = \left\{X \in \mathfrak{gl}(n, \mathbb{C}) : \bar{X}^T = -X\right\}. \tag{B.36}$$

[1]A subset I of a ring $(R, +, \cdot)$ is called an *ideal*, if $(I, +)$ is a subgroup of $(R, +)$ and $r \in R$, $x \in I$ implies $r \cdot x \in I$ and $x \cdot r \in I$.

(iii) *The special unitary groups.* Considering $SU(n)$ as a subgroup of $U(n)$ with determinant one and noting that $1 = \det(\exp(X)) = \exp(\mathrm{Tr}X)$, we get

$$\mathfrak{su}(n) = \left\{X \in \mathfrak{gl}(n, \mathbb{C}) : \bar{X}^T = -X \text{ and } \mathrm{Tr}X = 0\right\}. \tag{B.37}$$

(iv) *The symplectic groups.* Let J be the matrix in the definition of the symplectic group and either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. The condition for a matrix $X \in Mat(n, \mathbb{F})$ to be an element of $\mathfrak{sp}(n, \mathbb{F})$ is $\exp(tX)^T J \exp(tX) = J$ for all $t \in \mathbb{R}$. Differentiating this equation, evaluating it at $t = 0$ and using $J^2 = -I \in GL(2n, \mathbb{F})$ yields

$$\mathfrak{sp}(n, \mathbb{F}) = \left\{X \in \mathfrak{gl}(2n, \mathbb{F}) : JX^T J = X\right\}. \tag{B.38}$$

Further the Lie algebra of the compact symplectic group is given by

$$\mathfrak{sp}(n) = \mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(2n). \tag{B.39}$$

B.5 Invariant Forms on Lie Groups

Definition B.5.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(G, \mathfrak{g})$ with

$$\omega_g(X_g) := (\mathrm{d}L_{g^{-1}})(X_g) \in T_eG \cong \mathfrak{g}, \qquad \forall g \in G,\ \forall X_g \in T_gG \tag{B.40}$$

is called *canonical one-form* or *Maurer-Cartan form.*

Given a left-invariant vector field X on G, ω associates the generating vector X_e to X, i.e. i.e. ω describes the identification of T_eG with $\mathfrak{g}$.

Theorem B.5.1. *The Maurer-Cartan form ω satisfies*

$$L_g^*\omega = \omega, \tag{B.41}$$
$$R_g^*\omega = \mathrm{Ad}\,(g^{-1}) \circ \omega. \tag{B.42}$$

Proof. The left-invariance follows from the definition. For the second equation we take $v \in T_aG$, $g \in G$ and calculate:

$$\begin{aligned}(R_g^*\omega)_a(v) &= \omega_{ag}(\mathrm{d}R_g(v)) = \mathrm{d}L_{(ag)^{-1}}(\mathrm{d}R_g)(v) \\ &= \mathrm{d}L_{g^{-1}}(\mathrm{d}R_g(\mathrm{d}L_{a^{-1}}))(v) = \mathrm{Ad}\,(g^{-1})(\omega)_a(v).\end{aligned}$$

□

Theorem B.5.2. *The Maurer-Cartan form satisfies*

$$\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = 0, \tag{B.43}$$

which is known as the Maurer-Cartan equations*.*

Proof. Let X, Y be left-invariant vector fields on G. Then by the definition of the exterior derivative we have

$$\mathrm{d}\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - [\omega(X), \omega(Y)].$$

Since X and Y are left-invariant, we have $X(\omega(Y)) = Y(\omega(X))$ and the statement follows. □

Remark B.5.1. Let $n := \dim \mathfrak{g}$. We choose a basis $\{I_k\}_{k=1,\ldots,n}$ of $\mathfrak{g}$ with corresponding left-invariant vector fields $\{\tilde{I}_k\}_{k=1,\ldots,n}$, and one-forms ω^i on G determined by $\omega^i(\tilde{I}_j) = \delta^i{}_j$. Then (B.43) can be written as

$$\mathrm{d}\omega^k = -\frac{1}{2}\sum_{i,j=1}^{n} f_{ij}{}^k \omega^i \wedge \omega^j, \qquad k = 1, \ldots, n. \tag{B.44}$$

Definition B.5.2. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over some field $\mathbb{F}$. Then

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}, \qquad (X, Y) \mapsto \mathrm{Tr}(\mathrm{ad}\,(X) \circ \mathrm{ad}\,(Y)) \tag{B.45}$$

defines a symmetric bilinear form, where Tr is the trace of the endomorphism. B is then called the *Killing form* or *Cartan-Killing form* of $\mathfrak{g}$.

Remark B.5.2. Given a Lie group G with Lie algebra $\mathfrak{g}$, B is often called the Killing form of G.

Example B.5.1. (i) Let G be an abelian Lie group. Then the Cartan-Killing form is given by $B = 0$.

(ii) For the Lie algebra $\mathfrak{g} := \mathfrak{gl}(n, \mathbb{R})$ we find

$$B(X,Y) = 2n\,\mathrm{Tr}(X \circ Y) - 2\,\mathrm{Tr}(X) \cdot \mathrm{Tr}(Y). \tag{B.46}$$

(iii) Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra with $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{g}$ (i.e. $\mathfrak{h}$ is an ideal in $\mathfrak{g}$). We then have

$$B_{|\mathfrak{h}\times\mathfrak{h}} = B_{\mathfrak{h}}, \tag{B.47}$$

where $B := B_{\mathfrak{g}}$ denotes the Cartan-Killing form on $\mathfrak{g}$.

The Cartan-Killing form further has the following invariance properties.

Proposition B.5.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let B be the Cartan-Killing form of $\mathfrak{g}$. Let further $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an Lie algebra isomorphism. Then we have*

$$B(\sigma(X), \sigma(Y)) = B(X,Y), \qquad \forall X, Y \in \mathfrak{g}. \tag{B.48}$$

For the adjoint representation we find

$$B(\mathrm{Ad}\,(g)X, \mathrm{Ad}\,(g)Y) = B(X,Y), \tag{B.49}$$

$$B(\mathrm{ad}\,(X)(Y), Z) = -B(Y, \mathrm{ad}\,(X)(Z)), \tag{B.50}$$

for all $g \in G$ and $X, Y, Z \in \mathfrak{g}$.

One can proof the following

Theorem B.5.3. *A Lie algebra over a field* $\mathbb{F}$ *with* $\mathrm{char}\,\mathbb{F} = 0$ *is semisimple if and only if its Killing form is non-degenerate.*

Appendix C

Geometry of Principal Fibre Bundles

One way to construct geometric spaces from known ones is the Cartesian product. For two spaces M and G, their Cartesian product is denoted by $M \times G$. If the resulting space should incorporate a geometrically more complicated structure (like a twist in the Möbius strip), this concept does not suffice. Now one may construct a space which locally still looks like a Cartesian product, but globally has a more complicated structure. These spaces are called *fibre bundles*. This chapter is mainly based on [38, 41], and in parts on [48, 49, 51].

C.1 Principal Bundles and G-Structures

Definition C.1.1 (Principal Fibre Bundle)**.** Let M be a manifold and G a Lie group. A (*differentiable*) *principal fibre bundle over M with group G* consists of a manifold P and an action of G on P satisfying the following conditions:

(i) G acts freely on P on the right:

$$(u, a) \in P \times G \mapsto ua = R_a u \in P. \tag{C.1}$$

(ii) M is the quotient space of P by the equivalence relation induced by G, i.e. $M = P/G$, and the canonical projection $\pi : P \to M$ is differentiable.

(iii) P is locally trivial, that is, every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \phi(u))$ where ϕ is a mapping of $\pi^{-1}(U)$ into G satisfying $\phi(ua) = (\phi(u))a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

A principal fibre bundle will be denoted by $P(M, G, \pi)$, $P(M, G)$ or simply P. We call P the *total space* or *bundle space*, M the base space, G the *structure group* and π the projection. For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P, called the *fibre* over x. If u is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points ua, $a \in G$, and is called the fibre through u. Every fibre is diffeomorphic to G.

Given a Lie group G and a manifold M, G acts freely on $P := M \times G$ on the right as follows. For each $b \in G$, R_b maps $(x, a) \in M \times G$ into $(x, ab) \in M \times G$. The principal bundle $P(M, G)$ thus obtained is called *trivial*.

Definition C.1.2 (Associated Fibre Bundle)**.** Let $P(M, G)$ be a principal fibre bundle and F a manifold on which G acts on the left: $(a, \xi) \in G \times F \mapsto a\xi \in F$. On the product manifold $P \times F$, we let G act on the right as follows: An element $a \in G$ maps $(u, \xi) \in P \times F$ into $(ua, a^{-1}\xi) \in P \times F$. The quotient space of $P \times F$ by this group action is denoted by $E = P \times_G F$. The mapping $P \times F \to M$ which maps (u, x) into $\pi(u)$ induces a mapping π_E, called the projection, of E onto M. For each $x \in M$, the set $\pi_E^{-1}(x)$ is called the *fibre of E over x*. Every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. Identifying $\pi^{-1}(U)$ with $U \times G$, we see that the action of G on $\pi^{-1}(U) \times F$ on the right is given by

$$(x, a, \xi) \mapsto (x, ab, b^{-1}\xi), \qquad (x, a, \xi) \in U \times G \times F, \quad b \in G. \tag{C.2}$$

It follows that the isomophism $\pi^{-1}(U) \cong U \times G$ induces an isomorphism $\pi_E^{-1}(U) \cong U \times F$. We can therefore introduce a differentiable structure in E by the requirement that $\pi_E^{-1}(U)$ is an open submanifold of E which is diffeomorphic with $U \times F$ under the isomorphism $\pi_E^{-1}(U) \cong U \times F$. The projection π_E is then a differentiable mapping of E onto M. We call E (or more precisely $E(M, F, G, P)$) the *fibre bundle over M, with standard fibre F and structure group G, which is associated with the principal bundle P.*

Definition C.1.3 (Frame Bundle)**.** Let M be a manifold, and $E \to M$ a vector bundle with fibre $\mathbb{R}^k$. We define a manifold F^E by

$$F^E := \{(m, e_1, \ldots, e_k) : m \in M \text{ and } (e_1, \ldots, e_k) \text{ is a basis for } E_m\}. \tag{C.3}$$

We further define a projection onto the first component,

$$\pi : F^E \to M, \qquad (m, e_1, \ldots, e_k) \mapsto m. \tag{C.4}$$

For each $A = (A_{ij}) \in GL(k, \mathbb{R})$ and $(m, e_1, \ldots, e_k) \in F^E$, we define $A \cdot (m, e_1, \ldots, e_k) = (m, e_1', \ldots, e_k')$, where $e_i' := \sum_{j=1}^k A_{ij} e_j$. This gives an action of $GL(k, \mathbb{R})$ on F^E, which makes F^E into a principal bundle over M, with fibre $GL(k, \mathbb{R})$. We call F^E the *frame bundle* of E.

When $E = TM$, the bundle F^{TM} will be written F, and called the *frame bundle of M*.

Definition C.1.4. Let M be a manifold of dimension n, and F the frame bundle of M. Then F is a principal bundle over M with fibre $GL(n, \mathbb{R})$. Let G be a Lie subgroup of $GL(n, \mathbb{R})$. Then a *G-structure* on M is a principal subbundle P of F, with fibre G.

Example C.1.1. Let (M, g) be a Riemannian n-manifold, and F the frame bundle of M. Each point of F is $(x, e_1, \ldots, e_n)$, where $x \in M$ and $(e_1, \ldots, e_n)$ is a basis for $T_x M$. Define P to be a subset of F for which $(e_1, \ldots, e_n)$ is orthonormal with respect to g. Then P is a principal subbundle of F with fibre $O(n)$, so P is an $O(n)$-structure on M.

Definition C.1.5 (Adjoint Bundle)**.** Let P be a principal bundle over M with fibre G, let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathrm{ad} : G \to GL(\mathfrak{g})$ be the adjoint representation of

G on $\mathfrak{g}$. We may now construct a vector bundle $\mathrm{ad}\,(P)$ over M, with fibre $\mathfrak{g}$, called the *adjoint bundle*, in the following way.

Let ρ be a representation of G on a vector space V. Then G acts on the product space $P \times V$ by the principal bundle action on the first factor, and ρ on the second. Define $\rho(P) := (P \times V)/G$, the quotient of $P \times V$ by this G-action. Now $P/G = M$, so the obvious map from $(P \times V)/G$ to P/G yields a projection from $\rho(P)$ to M. Since G acts freely on P, this projection has fibre V, and thus $\rho(P)$ is a vector bundle over M, with fibre V.

Remark C.1.1. Let ρ be a representation of G on V, and $\pi : P \times V \to \rho(P)$ the natural projection. We may regard $P \times V$ as the trivial bundle over P with fibre V. Then if $e \in \mathscr{C}^\infty(\rho(P))$ is a smooth section of $\rho(P)$ over M, the pull-back $\pi^*(e)$ is a smooth section of $P \times V$ over P. Moreover, $\pi^*(e)$ is invariant under the action of G on $P \times V$, and this gives a 1-1 correspondence between sections of $\rho(P)$ over M and G-invariant sections $P \times V$ over P.

C.2 Holonomy Groups

As a geometrical consequence of the curvature of a connection, parallel-transporting geometrical objects around closed loops on a manifold will in general not map these objects onto itself. The holonomy of a connection measures the failure of the transported data to be preserved.

Let M be a manifold, E be a vector bundle over M and ∇^E the connection on E. Let $\gamma : [0,1] \to M$ be a smooth curve in M. Then the pull-back $\gamma^*(E)$ of E to $[0,1]$ is a vector bundle over $[0,1]$ with fibre $E_{\gamma(t)}$ over $t \in [0,1]$, where E_x is the fibre over $x \in M$.

Let s be a smooth section of $\gamma^*(E)$ over $[0,1]$, so that $s(t) \in E_{\gamma(t)}$ for each $t \in [0,1]$. The connection ∇^E pulls back under γ to give a connection $\gamma^*(E)$ over $[0,1]$. We say that s is *parallel* if its derivative under this pull-back connection is zero, i.e. if

$$\nabla^E_{\dot\gamma(t)} s(t) = 0 \qquad \forall t \in [0,1], \tag{C.5}$$

where $\dot\gamma(t)$ is $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$, regarded as a vector in $T_{\gamma(t)}M$.

Since this is a first-order ordinary differential equation in $s(t)$, for each possible initial value $e \in E_{\gamma(0)}$ there exists a unique, smooth solution s with $s(0) = e$. We shall use this to define the idea of *parallel transport* along γ.

Definition C.2.1. Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Suppose $\gamma : [0,1] \to M$ is smooth, with $\gamma(0) = x$ and $\gamma(1) = y$, where $x, y \in M$. Then for each $e \in E_x$, there exists a unique smooth section s of $\gamma^*(E)$ satisfying $\nabla^E_{\dot\gamma(t)} s(t) = 0$ for $t \in [0,1]$, with $s(0) = e$. Define $P_\gamma(e) = s(1)$. Then $P_\gamma : E_x \to E_y$ is a well-defined linear map, called the *parallel transport map*. This definition easily generalizes to the case when γ is continuous and piecewise-smooth, by requiring s to be continuous, and differentiable whenever γ is differentiable.

If we consider two paths α, β with $\alpha(0) = x$, $\alpha(1) = y = \beta(0)$, $\beta(1) = z$ with $x, y, z \in M$, then we define α^{-1} and $\beta\alpha$ by

$$\alpha^{-1}(t) = \alpha(1-t), \tag{C.6}$$

$$\beta\alpha(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} < t \le 1 \end{cases}. \tag{C.7}$$

Then α^{-1} and $\beta\alpha$ are piecewise-smooth paths in M with $\alpha^{-1}(0) = y$, $\alpha^{-1}(0) = x$, $\beta\alpha(0) = x$ and $\beta\alpha(1) = z$.

Suppose $e_x \in E_x$, and $P_\alpha(e_x) = e_y \in E_y$. Then there is a unique parallel section s of $\alpha^{-1}(E)$ with $s(0) = e_x$ and $s(1) = e_y$. Define $s'(t) = s(1-t)$. Then s' is a parallel section of $(\alpha^{-1})^*(E)$. Since $s'(0) = e_y$ and $s'(1) = e_x$, it follows that $P_{\alpha^{-1}}(e_y) = e_x$. Thus, if $P_\alpha(e_x) = e_y$, then $P_{\alpha^{-1}}(e_y) = e_x$, and so P_α and $P_{\alpha^{-1}}$ are inverse maps. In particular, this implies that if γ is any piecewise-smooth path in M, then P_γ is invertible. By a similar argument, we can also show that $P_{\beta\alpha} = P_\beta \circ P_\alpha$.

Definition C.2.2. Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Fix a point $x \in M$. We say that γ is a *loop (based at x)* if $\gamma : [0,1] \to M$ is a piecewise-smooth path with $\gamma(0) = \gamma(1) = x$. If γ is a loop based at x, then the parallel transport map $P_\gamma : E_x \to E_x$ is an invertible linear map, so that P_γ lies in the group of invertible linear transformations of E_x, denoted by $GL(E_x)$. We define the *holonomy group* $\mathrm{Hol}_x(\nabla^E)$ of ∇^E based at x to be

$$\mathrm{Hol}_x(\nabla^E) = \{P_\gamma : \gamma \text{ is a loop based at } x\} \subset GL(E_x). \tag{C.8}$$

If α, β are loops based at x, then the same hilds for α^{-1} and $\beta\alpha$. So, by the argument above, $\mathrm{Hol}_x(\nabla^E)$ is a subgroup of $GL(E_x)$.

The holonomy group is independent of the base-point x in the following sense:

Proposition C.2.1. *Let M be a (connected) manifold, E be a vector bundle over M with fibre $\mathbb{R}^k$, and ∇^E a connection on E. For each $x \in M$, the holonomy group $\mathrm{Hol}_x(\nabla^E)$ may be regarded as a subgroup of $GL(k,\mathbb{R})$, defined up to conjugation in $GL(k,\mathbb{R})$, and in this sense it is independent of the base point x.*

Proof. Suppose $x, y \in M$. Since M is connected, we can find a piecewise-smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$, so that $P_\gamma : E_x \to E_y$. If α is a loop based at x, then $\gamma\alpha\gamma^{-1}$ is a loop based at y, and $P_{\gamma\alpha\gamma^{-1}} = P_\gamma \circ P_\alpha \circ P_\gamma^{-1}$. Hence, if $P_\alpha \in \mathrm{Hol}_x(\nabla^E)$, then $P_\gamma \circ P_\alpha \circ P_\gamma^{-1} \in \mathrm{Hol}_y(\nabla^E)$. Thus

$$P_\gamma \mathrm{Hol}_x(\nabla^E) P_\gamma^{-1} = \mathrm{Hol}_y(\nabla^E). \tag{C.9}$$

This shows that the holonomy group $\mathrm{Hol}_x(\nabla^E)$ is independent of the base point x in the following sense. Suppose E has fibre $\mathbb{R}^k$. Then any identification $E_x \cong \mathbb{R}^k$ induces an isomorphism $GL(E_x) \cong GL(k,\mathbb{R})$, and so we may regard $\mathrm{Hol}_x(\nabla^E)$ as a subgroup H

of $GL(k, \mathbb{R})$. If we choose a different identification $E_x \cong \mathbb{R}^k$, we instead get the subgroup aHa^{-1} of $GL(k, \mathbb{R})$, for some $a \in GL(k, \mathbb{R})$. Thus, the holonomy group is a subgroup of $GL(k, \mathbb{R})$, defined up to conjugation. Moreover, (C.9) shows that if $x, y \in M$, then $\mathrm{Hol}_x(\nabla^E)$ and $\mathrm{Hol}_y(\nabla^E)$ yield the same subgroup of $GL(k, \mathbb{R})$, up to conjugation. $\square$

Because of this we may omit the subscript x and write the holonomy group of ∇^E as $\mathrm{Hol}(\nabla^E) \subset GL(k, \mathbb{R})$, implicitly supposing that two subgroups of $GL(k, \mathbb{R})$ are equivalent if they are conjugate in $GL(k, \mathbb{R})$. In the same way, if E is a complex vector bundle with fibre $\mathbb{C}^k$, then the holonomy group of ∇^E is a subgroup of $GL(k, \mathbb{C})$, up to conjugation. The proposition shows that the holonomy group is a *global invariant* of the connection.

Proposition C.2.2. *Let M be a simply-connected manifold, E a vector bundle over M with fibre $\mathbb{R}^k$, and ∇^E a connection on E. Then $\mathrm{Hol}(\nabla^E)$ is a connected Lie subgroup of $GL(k, \mathbb{R})$.*

Definition C.2.3. Let M be a manifold, E a vector bundle over M with fibre $\mathbb{R}^k$, and ∇^E a connection over E. Fix $x \in M$. A loop γ based at x is called *null-homotopic*, if it can be deformed to the constant loop at x. We define the *restricted holonomy group* $\mathrm{Hol}^0_x(\nabla^E)$ of ∇^E to be

$$\mathrm{Hol}^0_x(\nabla^E) = \{P_\gamma : \gamma \text{ is a null-homotopic loop based at } x\}. \tag{C.10}$$

Then $\mathrm{Hol}^0_x(\nabla^E)$ is a subgroup of $GL(E_x)$. As above we may regard $\mathrm{Hol}^0_x(\nabla^E)$ as a subgroup of $GL(k, \mathbb{R})$ defined up to conjugation, and it is independent of the base point x, and so is written $\mathrm{Hol}^0(\nabla^E) \subset GL(k, \mathbb{R})$.

We give some properties of $\mathrm{Hol}^0(\nabla^E)$.

Proposition C.2.3. *Let M be a manifold, E be a vector bundle over M with fibre $\mathbb{R}^k$, and ∇^E a connection on E. Then $\mathrm{Hol}^0(\nabla^E)$ is a connected Lie subgroup of $GL(k, \mathbb{R})$. It is the connected component of $\mathrm{Hol}(\nabla^E)$ containing the identity, and is a normal subgroup of $\mathrm{Hol}(\nabla^E)$. There is a natural, surjective group homomorphism[1] $\phi : \pi_1(M) \to \mathrm{Hol}(\nabla^E)/\mathrm{Hol}^0(\nabla^E)$. Thus, if M is simply-connected, then $\mathrm{Hol}(\nabla^E) = \mathrm{Hol}^0(\nabla^E)$.*

Definition C.2.4. Let M be a manifold, E a vector bundle over M with fibre $\mathbb{R}^k$, and ∇^E a connection on E. Then $\mathrm{Hol}^0(\nabla^E)$ is a Lie subgroup of $GL(k, \mathbb{R})$, defined up to conjugation. We define the *holonomy algebra* $\mathfrak{hol}(\nabla^E)$ to be the Lie algebra of $\mathrm{Hol}^0(\nabla^E)$. It is a Lie subalgebra of $\mathfrak{gl}(k, \mathbb{R})$, defined up to the adjoint action of $GL(k, \mathbb{R})$. Similarly $\mathrm{Hol}^0_x(\nabla^E)$ is a Lie subgroup of $GL(E_x)$ for all $x \in M$. Define $\mathfrak{hol}_x(\nabla^E)$ to be the Lie algebra of $\mathrm{Hol}^0_x(\nabla^E)$. It is a Lie subalgebra of $\mathrm{End}(E_x)$.

Finally we give a list of possible holonomy groups, which is known as *Berger's list.*

[1] $\pi_1(M)$ is the fundamental group of M, which is the group formed by the sets of equivalence classes of the set of all loops (i.e. paths with initial and final points at a given basepoint $p \in M$) under the equivalence relation of homotopy.

Theorem C.2.1 (Berger). *Suppose M is a simply-connected manifold of dimension n, and that g is a Riemannian metric on M, that is irreducible and nonsymmetric. Then exactly one of the following seven cases holds.*

(i) $\mathrm{Hol}(g) = SO(n)$,

(ii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = U(m)$ *in* $SO(2m)$,

(iii) $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = SU(m)$ *in* $SO(2m)$,

(iv) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = Sp(m)$ *in* $SO(4m)$,

(v) $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}(g) = Sp(m)Sp(1)$ *in* $SO(4m)$,

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $SO(7)$, *or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = Spin(7)$ *in* $SO(8)$.

A description of the groups $Spin(7)$ and G_2 is given in chapter 4.1.

We give a connection of Berger's list to the four division algebras in chapter A. First consider the cases (i)-(v). The group $SO(n)$ is a group of automorphisms of $\mathbb{R}^n$. Both $U(m)$ and $SU(m)$ are groups of automorphisms of $\mathbb{C}^m$, and $Sp(m)$ and $Sp(m)Sp(1)$ are automorphism groups of $\mathbb{H}^m$. To make the analogy between $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ more complete, we add the holonomy group $O(n)$. Then $O(n)$, $U(m)$ and $Sp(m)Sp(1)$ are automorphism groups of $\mathbb{R}^n$, $\mathbb{C}^m$ and $\mathbb{H}^m$ respectively, preserving a metric. $SO(n)$, $SU(m)$ and $Sp(m)$ are the subgroups of $O(n)$, $U(m)$ and $Sp(m)Sp(1)$ with "determinant 1" in an appropriate sense.

One can also regard G_2 and $Spin(7)$ as automorphism groups of $\mathbb{O}$. The octonions split as $\mathbb{O} \cong \mathbb{R} \oplus \mathrm{Im}\mathbb{O}$, where $\mathrm{Im}\mathbb{O} \cong \mathbb{R}^7$ is the *imaginary octionions*. The automorphism group of $\mathrm{Im}\mathbb{O}$ is G_2.

C.3 Connections on Principal Bundles

Suppose P is a principal bundle over a manifold M, with fibre G and projection $\pi : P \to M$. Let $p \in P$ and $m = \pi(p)$. Then the derivative of π gives a linear map $\mathrm{d}\pi_p : T_pP \to T_mM$. Define a subspace V_p of T_pP by $V_p := \mathrm{Ker}(\mathrm{d}\pi_p)$. Then the subspaces V_p form a vector subbundle V of the tangent bundle TP, called the *vertical subbundle*. Note that V_p is $T_p(\pi^{-1}(m))$, the tangent space to the fibre of $\pi : P \to M$ over m. But the fibres of π are the orbits of the free G-action on P. It follows that there is a natural isomorphism $V_p \cong \mathfrak{g}$ between V_p and the Lie algebra $\mathfrak{g}$ of G.

Definition C.3.1. Let M be a manifold, and P a principal bundle over M with fibre G, a Lie group. A *connection* on P is a vector subbundle D of TP called the *horizontal bundle*, that is invariant under the G-action on P, and which satisfies $T_pP = V_p \oplus D_p$ for each $p \in P$. If $\pi(p) = m$, then $\mathrm{d}\pi_p$ maps $T_pP = V_p \oplus D_p$ onto T_mM, and as $V_p = \mathrm{Ker}\mathrm{d}\pi_p$, we see that $\mathrm{d}\pi_p$ induces an isomorphism between D_p and T_mM.

Thus the horizontal subbundle D is naturally isomorphic to $\pi^*(TM)$. So if $v \in \mathfrak{X}(M)$ is a vector field on M, there is a unique section $\lambda(v)$ of the bundle $D \subset TP$ over P, such that $\mathrm{d}\pi_p(\lambda(v)_{|p}) = v_{\pi(p)}$ for each $p \in P$. We call $\lambda(v)$ the *horizontal lift* of v. It is a vector field on P, and is invariant under the action of G on P.

Remark C.3.1. Let M be a manifold of dimension n with frame bundle F, let $G \subset GL(n, \mathbb{R})$ be a Lie subgroup, and P a G-structure on M. Suppose D is a connection on P. Then there is a unique connection D' on F that reduces to D on P. Conversely, a connection D' on F reduces to a connection D on P if and only if for each $p \in P$, the subspace $D'_{|p}$ of T_pF lies in T_pP.

We now first define the curvature of a connection on a principal bundle and then relate the connection on a principal bundle to the connection on a vector bundle.

Let M be a manifold and P a principal bundle over M with fibre G, a Lie group with Lie algebra $\mathfrak{g}$, and D be a connection on P. If $v, w \in \mathscr{C}^\infty(TM)$ and α, β are smooth functions on M, then one can show that

$$[\lambda(\alpha v), \lambda(\beta w)] - \lambda([\alpha v, \beta w]) = \alpha\beta \cdot \{[\lambda(v), \lambda(w)] - \lambda([v, w])\}, \tag{C.11}$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. Thus the expression $[\lambda(v), \lambda(w)] - \lambda([v, w])$ is pointwise-linear and antisymmetric in v, w. Also, as $\mathrm{d}\pi(\lambda(v)) = v$ for all vector fields v on M we see that

$$\mathrm{d}\pi([\lambda(v), \lambda(w)]) = \mathrm{d}\pi(\lambda([v, w])) = [v, w]. \tag{C.12}$$

Therefore, $[\lambda(v), \lambda(w)] - \lambda([v, w])$ lies in the kernel of $\mathrm{d}\pi$, which is the vertical subbundle C of TP. Since there is a natural isomorphism $C_p \cong \mathfrak{g}$ for each $p \in P$, we may regard $[\lambda(v), \lambda(w)] - \lambda([v, w])$ as a section of the trivial vector bundle $P \times \mathfrak{g}$ over P.

As $\lambda(v), \lambda(w)$ and $\lambda([v, w])$ are invariant under the action of G on P, this section of $P \times \mathfrak{g}$ is invariant under the natural action of G on $P \times \mathfrak{g}$. But from remark C.1.1 there is a 1-1 correspondence between G-invariant sections of $P \times \mathfrak{g}$ over P, and sections of the adjoint bundle $\mathrm{ad}\,(P)$ over M. We use this to deduce the following result, which defines the curvature $R(P, D)$ of a connection D on P.

Proposition C.3.1. *Let M be a manifold, G a Lie group with Lie algebra $\mathfrak{g}$, P a principal bundle over M with fibre G, and D a connection on P. Then there exists a unique, smooth section $R(P, D)$ of the bundle* $\mathrm{ad}\,(P) \otimes \Omega^2(M)$ *called the* curvature of D, *that satisfies*

$$\pi^*(R(P, D) \cdot v \wedge w) = [\lambda(v), \lambda(w)] - \lambda([v, w]) \tag{C.13}$$

for all $v, w \in \mathscr{C}^\infty(TM)$. Here the left hand side is a $\mathfrak{g}$-valued function on P, the right hand side is a section of the subbundle $C \subset TP$, and the two sides are identified using the natural isomorphism $C_p \cong \mathfrak{g}$ for $p \in P$.

Next we relate connections on vector and principal bundles. For this, let D be a connection on a principal bundle P with M and G as in the proposition above. We then have for each p the isomorphisms

$$T_pP \cong C_p \oplus D_p, \quad C_p \cong \mathfrak{g} \quad \text{and} \quad \mathrm{D}_p \cong \pi^*(T_{\pi(p)}M).$$

These give a natural splitting

$$V \otimes T^*P \cong (V \otimes \mathfrak{g}^*) \oplus (V \otimes \pi^*(T^*M)),$$

where $e \in \mathscr{C}^\infty(E)$, so that $\pi^*(e)$ is a section of $P \times V$ over P. $\pi^*(e)$ is a function $\pi^*(e) : P \to V$, so its exterior derivative is a linear map $\mathrm{d}\pi^*(e)_{|p} : T_pP \to V$ for each $p \in P$. Thus $\mathrm{d}\pi^*(e)$ is a smooth section of the vector bundle $V \otimes T^*P$ over P. Write $\pi_D(\mathrm{d}\pi^*(e))$ for the component of $\mathrm{d}\pi(e)$ in $\mathscr{C}^\infty(V \otimes \pi^*(T^*M))$ in this splitting. Now both $\pi^*(e)$ and the vector bundle splitting are G-invariant, so $\pi_D(\mathrm{d}\pi^*(e))$ must be G-invariant. Since there is a 1-1 correspondence between G-invariant sections of $V \otimes \pi^*(T^*M)$ over P, and sections of the corresponding vector bundle $E \otimes T^*M$ over M, $\pi_D(\mathrm{d}\pi^*(e))$ is the pull-back of a unique element of $\mathscr{C}^\infty(E \otimes T^*M)$. We use this to define ∇^E.

Definition C.3.2. Let M be a manifold, P a principal bundle over M with fibre G, and D a connection on P. Let ρ be a representation of G on a vector space V, and define E to be the vector bundle $\rho(P)$ over M. If $e \in \mathscr{C}^\infty(E)$, then $\pi_D(\mathrm{d}\pi^*(e))$ is a G-invariant section of $V \otimes \pi^*(T^*M)$ over P. Define $\nabla^E e \in \mathscr{C}^\infty(E \otimes T^*M)$ to be the unique section of $E \otimes T^*M$ with pull-back $\pi_D(\mathrm{d}\pi^*(e))$ under the natural projection $V \otimes \pi^*(T^*M) \to E$. This defines a connection ∇^E on the vector bundle E over M.

To each connection D on a principal bundle P, we have associated a unique connection ∇^E on the vector bundle $E = \rho(P)$. If $G = GL(k, \mathbb{R})$ and ρ is the standard representation of G on $\mathbb{R}^k$, so that P is the frame bundle F^E of E, then this gives a 1-1 correspondence between connections on P and E.

Finally we relate the ideas of curvature in vector and principal bundles.

Proposition C.3.2. *Suppose M is a manifold, G a Lie group with Lie algebra $\mathfrak{g}$, P a principal bundle over M with fibre G, and D a connection on P, with curvature $R(P,D)$. Let ρ be a representation of G on a vector space V, E the vector bundle $\rho(P)$ over M, and ∇^E the connection in the previous definition, with curvature $R(\nabla^E)$.*

Now $\mathfrak{g}$ and $\mathrm{End}(V)$ are representations of G, and ρ gives a G-equivariant linear map $\mathrm{d}\rho : \mathfrak{g} \to \mathrm{End}(V)$. This induces a map $\mathrm{d}\rho : \mathrm{ad}\,(P) \to \mathrm{End}(E)$ of the vector bundles $\mathrm{ad}\,(P)$ and $\mathrm{End}(E)$ over M corresponding to $\mathfrak{g}$ and $\mathrm{End}(V)$. Let

$$\mathrm{d}\rho \otimes \mathrm{id} : \mathrm{ad}\,(P) \otimes \Omega^2(M) \to \mathrm{End}(E) \otimes \Omega^2(M)$$

be the product with the identity on $\Omega^2(M)$. Then $(\mathrm{d}\rho \otimes \mathrm{id})(R(P,D)) = R(\nabla^E)$.

Thus, the definitions of curvature of connections in vector and principal bundles are essentially equivalent.

Another (but related) description of the curvature on a principal bundle is given by the curvature form. In order to write down its definition, we begin by introducing the so-called connection form, which in the same manner may be thought of as an alternative to a connection.

Definition C.3.3 (Connection Form)**.** Let $P(M, G)$ be a principal bundle and $\omega \in \Omega^1(P, \mathfrak{g})$. We call ω a *connection form*, if the following two conditions are satisfied.

(i) $R_g^*\omega = \mathrm{Ad}\,(g^{-1}) \circ \omega$, for all $g \in G$.

(ii) $\omega(\widetilde{X}) = X$, for all $X \in \mathfrak{g}$.

We label the set of all connection forms on P by $\mathcal{C}(P)$.

Now one can show that we have a 1-1 correspondence between connections and connection forms on principal bundles, namely:

Proposition C.3.3. *Let $P(M, G)$ be a principal bundle. Then there is a 1-1 correspondence between connections and connection forms:*

(i) Let $D_p \subset T_pP$ be a horizontal subbundle at $p \in P$. We then define a connection form by

$$\omega_p(\widetilde{X}(p) \oplus Y) := X, \qquad \forall p \in P, X \in \mathfrak{g}, Y \in D_p. \tag{C.14}$$

(ii) Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection form. Then we define a connection on P by

$$D : p \in P \mapsto D_pP := \ker \omega_p. \tag{C.15}$$

Remark C.3.2. The connection form ω is also known as the *Ehresmann connection.*

Remark C.3.3 (Local Connection Form)**.** Let ω be a connection form on a principal bundle $P(M, G)$. Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of M for some index set $I \subset \mathbb{N}$, and let $U \in \{U_\alpha\}_{\alpha \in I}$. We choose a local section

$$s \in \Gamma(U, \pi^{-1}(U)), \tag{C.16}$$

where $\pi : P \to M$ is the projection in $P(M, G)$ (cf. definition C.1.1), and define

$$\mathcal{A}_s := s^*\omega \in \Omega^1(U, \mathfrak{g}). \tag{C.17}$$

On the other hand, if we have given a connection form $\mathcal{A} \in \Omega^1(U, \mathfrak{g})$ on a patch $U \in \{U_\alpha\}_{\alpha \in I}$ together with a local section $s \in \Gamma(U, \pi^{-1}(U))$, then there exists a unique connection form ω on $\pi^{-1}(U)$ such that $\mathcal{A}_s = s^*\omega$. Explicitly, this is given by the expression

$$\omega_{|U} = g^{-1}\pi^*\mathcal{A}_s g + g^{-1}\mathrm{d}_p g, \tag{C.18}$$

where d_p denotes the exterior derivative on P and g is defined via the homeomorphism

$$\phi : \pi^{-1}(U) \to U \times G, \quad u \mapsto (p, g), \tag{C.19}$$

thus

$$\phi(u) = (p, g), \quad \text{with} \quad u = s(p)g. \tag{C.20}$$

Remark C.3.4 (Transformation Property of the Connection Form)**.** We choose the same setting as in the previous remark. Let U and V be two overlapping neighborhoods and let

$$\omega_{|U} = \omega_{|V} \quad \text{on} \quad U \cap V \neq 0. \tag{C.21}$$

From (C.18) we get the condition

$$g_s^{-1}\pi^*\mathcal{A}_s g_s + g_s^{-1}\mathrm{d}_p g_s = g_t^{-1}\pi^*\mathcal{A}_t g_t + g_t^{-1}\mathrm{d}_p g_t, \tag{C.22}$$

with $s \in \Gamma(U, \pi^{-1}(U))$, $t \in \Gamma(U, \pi^{-1}(V))$, and g_s, g_t defined analogous to g in the previous remark. We define the *transition functions* ϕ_{st} by

$$g_s = \phi_{st}(p)g_t, \qquad \text{i.e.} \quad t(p) = s(p)\phi_{st}(p), \qquad p \in U \cap V. \tag{C.23}$$

Inserting this into equation (C.22), we get

$$\pi^*\mathcal{A}_t = \phi_{st}^{-1}\pi^*\mathcal{A}_s\phi_{st} + \phi_{st}^{-1}\mathrm{d}_p\phi_{st}. \tag{C.24}$$

Therefore, we have the following transformation behavior on the base manifold M:

$$\mathcal{A}_t = \phi_{st}^{-1}\mathcal{A}_s\phi_{st} + \phi_{st}^{-1}\mathrm{d}\phi_{st}. \tag{C.25}$$

Remark C.3.5. A nontrivial principal bundle does not admit a global section, so the pullback $\mathcal{A}_s = s^*\omega$ exists locally but not globally. Therefore we have at least two different local connection forms, which are related by the transformation condition (C.25).

We now introduce the notion of the exterior covariant derivative and the curvature form.

Definition C.3.4. Let $D : \Omega^k(P, V) \to \Omega^{k+1}(P, V)$ be the linear map with

$$(D\omega)_p(X_0, \ldots, X_k) := (\mathrm{d}_p\omega)(X_0^H, \ldots, X_k^H), \quad X_0, \ldots, X_k \in T_pP, \tag{C.26}$$

where $X_i^H \in D_pP$ are the horizontal components of X_i (cf. definition C.3.1) and d_p denotes the exterior derivative on P. The map D is called the *exterior covariant derivative.*

Definition C.3.5 (Curvature Form)**.** Let $P(M, G)$ be a principal bundle and $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection form. Then we define the *curvature form* (which is associated to ω) by

$$\Omega := D\omega \in \Omega^2(P, \mathfrak{g}). \tag{C.27}$$

Definition C.3.6 (Canonical One-Form)**.** Let P be a principal bundle, $u \in P$, and let F be the frame bundle. The *canonical one-form* θ of F is the $\mathbb{R}^n$-valued 1-form on F defined by

$$\theta(X) = u^{-1}(\mathrm{d}\pi(X)), \qquad X \in T_u F, \tag{C.28}$$

where u is considered as a linear mapping of $\mathbb{R}^n$ onto $T_{\pi(u)}M$, and $\pi : F \to M$ denotes the projection.

Definition C.3.7 (Linear Connection)**.** A connection in the bundle of linear frames F over M is called a *linear connection* of M.

Definition C.3.8 (Torsion Form)**.** We then define the *torsion form* by

$$T := D\theta. \tag{C.29}$$

Theorem C.3.1 (Structure Equations)**.** *Let ω, T and Ω be the connection form, the torsion form and the curvature form of a linear connection on M. Then ω, T and Ω satisfy* Cartan's structure equations, *which are given by*

$$T(X,Y) = \mathrm{d}\theta(X,Y) + \frac{1}{2}\left(\omega(X)\theta(Y) - \omega(Y)\theta(X)\right), \tag{C.30}$$

$$\Omega(X,Y) = \mathrm{d}\omega(X,Y) + \frac{1}{2}[\omega(X), \omega(Y)]. \tag{C.31}$$

These equations are also called the first and second structure equations*, respectively. The curvature form Ω further satisfies*

$$R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega = g^{-1}\Omega g, \qquad \forall g \in G. \tag{C.32}$$

and the Bianchi identity $D\Omega = 0$.

Remark C.3.6 (Coordinate Representation)**.** With respect to the natural basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, we write

$$\theta = \sum_{\alpha=1}^{n} \theta^\alpha e_\alpha \quad \text{and} \quad T = \sum_{\alpha=1}^{n} T^\alpha e_\alpha. \tag{C.33}$$

Let ${E_\alpha}^\beta$ be the basis of $\mathfrak{gl}(n,\mathbb{R})$ where ${E_\alpha}^\beta$ denotes the $n \times n$-matrix such that the entry at the α-th column and the β-th row is 1 and all others are zero. Then with respect to ${E_\alpha}^\beta$ we write

$$\omega = \sum_{\alpha,\beta=1}^{m} {\omega^\alpha}_\beta {E_\alpha}^\beta. \tag{C.34}$$

Now the structure equations can be written as

$$T^\alpha = \mathrm{d}\theta^\alpha + \sum_{\beta=1}^{m} {\omega^\alpha}_\beta \wedge \theta^\beta, \qquad \alpha = 1, \ldots, m, \tag{C.35}$$

$${\Omega^\alpha}_\beta = \mathrm{d}{\omega^\alpha}_\beta + \sum_{\gamma=1}^{m} {\omega^\alpha}_\gamma \wedge {\omega^\gamma}_\beta, \qquad \alpha, \beta = 1, \ldots, m. \tag{C.36}$$

Remark C.3.7. Let $P(M,G)$ be a principal bundle and let $F := P \times_G V$ be the associated bundle, and let ρ be a representation of G on V. Consider a section $g \in \Gamma(M,F)$. Then the covariant derivative is given by

$$Dg = \mathrm{d}g + \rho(\mathcal{A})g. \tag{C.37}$$

Especially for the adjoint bundle $\mathrm{ad}\,(P) \cong P \times \mathfrak{g}$ we get

$$Dg = \mathrm{d}g + \mathrm{ad}\,(\mathcal{A}) \circ g. \tag{C.38}$$

To close this chapter, we write down a localized version of the curvature form.

Definition C.3.9 (Local Curvature Form)**.** Let Ω be the curvature form of a principal bundle $P(M,G)$. Then we define the *local curvature form*

$$\mathcal{F} := s^*\Omega \in \Omega^2(U,\mathfrak{g}), \tag{C.39}$$

where s^* is the pullback of a local section $s \in \Gamma(U,\pi^{-1}(U))$ on a chart $U \subset M$.

Remark C.3.8. With $\mathcal{A} := s^*\omega$ we also find

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}. \tag{C.40}$$

In terms of components, we have

$$\mathcal{A} = \sum_{\mu=1}^{n} \mathcal{A}_\mu \mathrm{d}x^\mu, \qquad \mathcal{F} = \frac{1}{2}\sum_{\mu,\nu=1}^{n} \mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{C.41}$$

with

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu,\mathcal{A}_\nu]. \tag{C.42}$$

Example C.3.1 (The Curvature Form of the Tangent Bundle)**.** Consider the tangent bundle TM of a Riemannian manifold M. It has structure group $O(n)$, so the curvature form Ω is a 2-form with values in $\mathfrak{o}(n)$, which are the antisymmetric $n \times n$ real matrices. Then for $X,Y \in TM$ we have $R(X,Y) = \Omega(X,Y)$, so the curvature form is an alternative description of the Riemannian curvature tensor R.

Definition C.3.10 (Spin Connection Form)**.** Let M be a manifold of dimension m and F be the tangent frame bundle. Let $U \subset M$ be an open neighborhood and $s \in \Gamma(U,\pi^{-1}(U))$ a local section, where $\pi : F \to M$ denotes the projection. If $\omega \in \Omega^1(F,\mathfrak{so}(m))$ with $s^*\omega \in \Omega^1(U,\mathfrak{so}(m))$, we call $s^*\omega$ a *spin connection form.*

Remark C.3.9. (i) The spin connection form $s^*\omega$ may be regarded as a $m\times m$-matrix of one-forms.

(ii) We consider a frame $e \in F$ with a local basis $\{e_\alpha\}_{\alpha=1}^m$ at $p \in M$. Let $\omega^\alpha{}_\beta$ denote the components of the spin connection form, i.e. a matrix of one forms, such that

$$De_\alpha = \sum_{\beta=1}^{m} e_\beta \otimes \omega^\beta{}_\alpha, \tag{C.43}$$

where D denotes the exterior covariant derivative. For $\xi := \sum_{\alpha=1}^m e_\alpha \xi^\alpha$ we then get

$$\sum_{\alpha=1}^{m} D(e_\alpha \xi^\alpha) = \sum_{\alpha=1}^{m} e_\alpha \otimes \mathrm{d}\xi^\alpha + \sum_{\alpha,\beta=1}^{m} e_\beta \otimes \omega^\beta{}_\alpha \xi^\alpha, \tag{C.44}$$

which may be written as $D\xi = (\mathrm{d} + \omega)\xi$.

Appendix D

Coset Space Representations

We state the structure constants and some matrix representations of the coset spaces used in this thesis. For further references, see [16] or [17]. Note that the structure constants we use are rescaled with a factor of $1/\sqrt{3}$ with respect to these two papers.

D.1 $G_2/SU(3)$

The structure constants are given by

$$\begin{aligned}
f_{163} = f_{145} = f_{253} = f_{264} &= \frac{1}{\sqrt{3}} \\
f_{736} = f_{745} = f_{853} = f_{846} &= f_{956} = f_{10,16} = f_{10,52} \\
&= f_{11,51} = f_{11,52} = f_{12,41} = f_{12,32} = f_{13,31} = f_{13,24} = \frac{1}{2} \\
f_{14,43} = f_{14,56} &= \frac{1}{2\sqrt{3}}, \\
f_{14,21} &= \frac{1}{\sqrt{3}}, \\
f_{i+6,j+6,k+6} &= f_{\mathrm{GM}ijk},
\end{aligned} \tag{D.1}$$

where $f_{\mathrm{GM}ijk}$ are the Gell-Mann structure constants.

D.2 $SU(3)/U(1) \times U(1)$

The coset space $SU(3)/U(1) \times U(1)$ may be generated with the following (rescaled) Gell-Mann matrices.

$$I_1 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I_2 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{D.2}$$

$$I_3 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad I_4 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \tag{D.3}$$

$$I_5 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad I_6 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \tag{D.4}$$

$$I_7 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I_8 = \frac{-\mathrm{i}}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{D.5}$$

The structure constants for the corresponding Lie algebra then are given by

$$\begin{aligned} f_{127} &= \frac{1}{\sqrt{3}}, \\ f_{136} &= -f_{145} = f_{235} = f_{246} = f_{347} = -f_{567} = \frac{1}{2\sqrt{3}}, \\ f_{348} &= f_{568} = \frac{1}{2}. \end{aligned} \tag{D.6}$$

The $U(1) \times U(1)$-subgroup is embedded via the two diagonal generators I_7 and I_8.

D.3 $Sp(2)/Sp(1) \times U(1)$

The coset space $Sp(2)/Sp(1) \times U(1)$ may be generated with the following matrices (4-dimensional fundamental representation).

$$I_1 = \frac{-\mathrm{i}}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad I_2 = \frac{-\mathrm{i}}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \tag{D.7}$$

$$I_3 = \frac{-\mathrm{i}}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad I_4 = \frac{-\mathrm{i}}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{D.8}$$

$$I_5 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad I_6 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{D.9}$$

$$I_7 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad I_8 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{D.10}$$

$$I_9 = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad I_{10} = \frac{-\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{D.11}$$

The structure constants for the corresponding Lie algebra are then given by

$$
\begin{aligned}
&f_{541} = f_{532} = f_{613} = f_{642} = \frac{1}{2\sqrt{3}}, \qquad f_{756} = f_{10,89} = -\frac{1}{\sqrt{3}} \\
&f_{721} = f_{743} = f_{814} = f_{832} = f_{913} = f_{924} = f_{10,34} = f_{10,21} = \frac{1}{2\sqrt{3}}
\end{aligned} \tag{D.12}
$$

The $Sp(1) \times U(1)$-subgroup is embedded via the generators I_7, I_8, I_9, I_{10}.

Appendix E

The Jacobi Elliptic Functions

The Jacobi elliptic functions arise from the inversion of the elliptic integral of the first kind,

$$u = F(\xi, k) = \int_0^\xi \frac{\mathrm{d}t}{\sqrt{1 - k^2 \sin t}}, \qquad 0 \leq k^2 < 1, \tag{E.1}$$

where $k = \operatorname{mod} u$ is the elliptic modulus and $\xi = \operatorname{am}(u, k) = \operatorname{am}(u)$ is the Jacobi amplitude, giving

$$\xi = F^{-1}(u, k) = \operatorname{am}(u, k). \tag{E.2}$$

Then the three basic functions sn, cn and dn are defined by

$$\operatorname{sn}[u, k] = \sin(\operatorname{am}(u, k)) = \sin \xi, \tag{E.3}$$
$$\operatorname{cn}[u, k] = \cos(\operatorname{am}(u, k)) = \cos \xi, \tag{E.4}$$
$$\operatorname{dn}[u, k]^2 = 1 - k^2 \sin^2(\operatorname{am}(u, k)) = 1 - k^2 \sin^2 \xi. \tag{E.5}$$

These functions are periodic in $K(k)$ and $\tilde{K}(k)$,

$$\operatorname{sn}[u + 2mK + 2n\mathrm{i}\tilde{K}, k] = (-1)^m \operatorname{sn}[u, k], \tag{E.6}$$
$$\operatorname{cn}[u + 2mK + 2n\mathrm{i}\tilde{K}, k] = (-1)^{m+n} \operatorname{cn}[u, k], \tag{E.7}$$
$$\operatorname{dn}[u + 2mK + 2n\mathrm{i}\tilde{K}, k] = (-1)^n \operatorname{dn}[u, k], \tag{E.8}$$

where $K(k)$ is the complete elliptic integral of the first kind,

$$K(k) := F\left(\frac{\pi}{2}, k\right) \quad \text{and} \quad \tilde{K}(k) := K\left(\sqrt{1-k^2}\right) = F\left(\frac{\pi}{2}, \sqrt{1-k^2}\right). \tag{E.9}$$

In the following we sometimes drop the parameter k, i.e. $\operatorname{sn}[u; k] = \operatorname{sn}(u)$ etc.

The Jacobi elliptic functions generalize the trigonometric functions and satisfy analogous identities, including

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \tag{E.10}$$
$$k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1, \tag{E.11}$$
$$\operatorname{cn}^2 u + \sqrt{1-k^2}\, \operatorname{sn}^2 u = 1 \tag{E.12}$$

as well as

$$\mathrm{sn}(u,0) = \sin u, \tag{E.13}$$
$$\mathrm{cn}(u,0) = \cos u, \tag{E.14}$$
$$\mathrm{dn}(u,0) = 1. \tag{E.15}$$

One may also define cn, dn and sn as solutions $y(x)$ to the differential equations

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = (2-k)^2 y + y^3, \tag{E.16}$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -(1-2k^2)y + 2k^2 y^3, \tag{E.17}$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -(1+k^2)y + 2k^2 y^3. \tag{E.18}$$

Appendix F

Mathematica Source Code

This is the source code which essentially did the numerical simulations used in this thesis. Note that the potential has to be inverted for the dyonic solutions. To create the plots in chapter 7, the values in the tables F.1 and F.2 were used.

```
PotTerm[p_,k_] := (k-1)*p - (k+3)*Conjugate[p]^2 + 4*Abs[p]^2*p;
Equations[k_,pRe_,pIm_,t_]:={ 6*pRe''[t]==Re[PotTerm[k,pRe[t]+I*pIm[t]]],
  6*pIm''[t]==Im[PotTerm[k,pRe[t]+I*pIm[t]]] };
NumSol:=NDSolve[Join[Equations[k,pRe,pIm,t],
  ODEConditions[pRe,pIm,tmin,tmax]], {pRe,pIm}, {t,tmin,tmax},
  Method->"ExplicitRungeKutta"];
V[k_,p_]:=((1-k/3) + (k-1)*Abs[p]^2-(1+k/3)*(p^3+Conjugate[p]^3)
  + 2*Abs[p]^4);
VPlot:=ContourPlot[V[x+I*y,k],{x,-1.1,1.1},{y,-1.1,1.1},
  ColorFunction->"Pastel", Contours->{0, 1/30, 1/10, 2/10, 3/10, 4/10,
  5/10, 6/10, 7/10, 8/10, 9/10, 125/128, 190/200, 197/200, 399/400,
  16/15, 12/10, 14/10, 16/10, 18/10, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6,
  6.5, 7}];
NumSolPlot:=
  ParametricPlot[Evaluate[{{pRe[t],pIm[t]},
    {Re[Exp[I*2*Pi/3]*(pRe[t]+I*pIm[t])],
    Im[Exp[I*2*Pi/2]*(pRe[t]+I*pIm[t])]},
    {Re[Exp[I*4*Pi/3]*(pRe[t]+I*pIm[t])],
    Im[Exp[I*4*Pi/2]*(pRe[t]+I*pIm[t])]} } /.  NumSol],
  {t, tmin, tmax}, Mesh->Full, PlotRange->1.1, PlotStyle->{Blue}];
PointSet:={Red, PointSize[Large], Point[{0,0}],
  Point[{1,0}], Point[{Re[Exp[I*2*Pi/3]],Im[Exp[I*2*Pi/3]]}],
  Point[{Re[Exp[I*4*Pi/3]],Im[Exp[I*4*Pi/3]]}], Point[{1/4*k-1/4,0}],
  Point[{(1/4*k-1/4)*Re[Exp[I*2*Pi/3]],(1/4*k-1/4)*Im[Exp[I*2*Pi/3]]}],
  Point[{(1/4*k-1/4)*Re[Exp[I*4*Pi/3]],(1/4*k-1/4)*Im[Exp[I*4*Pi/3]]}]};
InitialVelocity := 10^(-5);
InitialAngle:=1.05581637352150657;
ODEConditions[pRe_,pIm_,tmin_,tmax_]:={pRe[tmin]==Re[Exp[I*4*Pi/3]],
  pIm[tmin]==Im[Exp[I*4*Pi/3]],
  pRe'[tmin]==InitialVelocity*Re[Exp[I*InitialAngle]],
```

```
  pIm´[tmin]==InitialVelocity*Im[Exp[I*InitialAngle]]};
Block[{k=0,tmin=0,tmax=33},Show[VPlot,NumSolPlot],
  Epilog->{PointSet,
    Inset[Framed[Style["\[kappa]=0",20],Background->LightYellow],
    Scaled[{0.8,0.1}]]}]];
```

In order to compute and plot the total energy of the solution, the following line may be added.

```
Block[{k=0,tmin=0,tmax=33},
  Plot[Evaluate[((pRe´[t]^2+pIm´[t]^2-1/6*V[pRe[t]+I*pIm[t]],k]) /.
    NumSol], {t,tmin,tmax}, PlotRange->{{0,tmax},{0.01,0.01}}]]
```

κ	`InitialVelocity`	`InitialAngle`	`tmax`	Starting Point
-4	10^{-2}	2.6093328657958325	20	$\exp(4\pi\mathrm{i}/3)$
-3	10^{-5}	$\pi/3$	22	$\exp(4\pi\mathrm{i}/3)$
-2	10^{-5}	2.61233791818	31	$\exp(4\pi\mathrm{i}/3)$
-1	10^{-5}	$\pi/2$	33	$\exp(4\pi\mathrm{i}/3)$
0	10^{-5}	1.05581637352150657	33	$\exp(4\pi\mathrm{i}/3)$
1	10^{-5}	1.047262330677851	29	$\exp(4\pi\mathrm{i}/3)$
2	10^{-5}	1.04719764726756	27	$\exp(4\pi\mathrm{i}/3)$
2.5	10^{-5}	1.04719755211932065	33	$\exp(4\pi\mathrm{i}/3)$
3	10^{-5}	π	33	1
4	0.1	π	12	1

Table F.1: Parameters for the numerical instanton solutions. `tmin` is always 0.

κ	`InitialVelocity`	`InitialAngle`	`tmax`	Starting Point
$-61/3$	10^{-5}	$\pi/2$	10	1
-7	10^{-5}	$\pi/2$	30	1
-3	10^{-5}	$\pi/3$	60	0
-1.7989	10^{-5}	$\pi/2$	100	$\frac{1}{4}(\kappa-1)$
-1	10^{-5}	$\pi/2$	33	$\frac{1}{4}(\kappa-1)$
0	10^{-5}	$\pi/2$	30	$\frac{1}{4}(\kappa-1)$
0.25	10^{-5}	$\pi/2$	50	$\frac{1}{4}(\kappa-1)$
0.5	10^{-5}	$\pi/2$	50	$\frac{1}{4}(\kappa-1)$
1	10^{-5}	$2\pi\mathrm{i}/3$	200	0
2	10^{-5}	0	33	$\frac{1}{4}(\kappa-1)$
4	10^{-5}	0	35	$\frac{1}{4}(\kappa-1)$
9	10^{-5}	0	35	1

Table F.2: Parameters for the numerical dyon solutions. `tmin` is always 0.

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